HOMEWORK 1

YOUR NAME

Due date: Monday of Week 5

Problem 1. *Let $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Consider the relation R on X defined by*

$$R = \{((a,b),(c,d)) \in X \times X : ad = bc\}.$$

Show that R is an equivalence relation. (Actually X/R is the set of rational numbers. Think about why this is so.)

Problem 2. *Show that the set $F = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$ with the usual addition and multiplication is a field.*

Problem 3. *In this problem, we construct a field $(F, +, \times)$ which has 4 elements. Denote the 4 elements by $\{0, 1, \alpha, \beta\}$, where 0 is the identity of the abelian group $(F, +)$, 1 is the identity element of the abelian group $(F - \{0\}, \times)$, α, β are the other two elements in F. Fill the following tables of the binary operations and check that F together with these two binary operations is indeed a field. (You only need to write up the details that every element in $F - \{0\}$ has an inverse.)*

$+$	0	1	α	β
0				
1				
α				
β				

$\times$	0	1	α	β
0				
1				
α				
β				

Problem 4. *In this problem, we construct a field $(F, +, \cdot)$ which has 9 elements. Let $\mathbb{F}_3 = \{0, 1, 2\}$ be the finite field with 3 elements with the usual notation. This means that 0 is the additive identity, 1 is the multiplicative identity and $2 = 1 + 1$. Let α be an extra element, which is not in $\mathbb{F}_3$ but $\alpha^2 = 2$ (where the 2 in α^2 means $\alpha^2 = \alpha \cdot \alpha$ and 2 on the right side denotes the element "2" in $\mathbb{F}_3$). Consider the set $F = \{a\alpha + b | a, b \in \mathbb{F}_3\}$. Define addition and multiplication on F in the usual sense. Namely,*

$$(a_1\alpha + b_1) + (a_2\alpha + b_2) = (a_1 + a_2)\alpha + (a_2 + b_2),$$
$$(a_1\alpha + b_1) \cdot (a_2\alpha + b_2) = (a_1b_2 + a_2b_1)\alpha + (2a_1a_2 + b_1b_2),$$

for $a_1, a_2, b_1, b_2 \in \mathbb{F}_3$. Show that $(F, +, \cdot)$ is a field. This is a field with 9 elements.

(Using similar method, try to construct a field with 25 elements.)