HOMEWORK 4

NAME:

Due date: Monday of Week 9

Exercises: 4, 7, (page 34)
Exercises: 9, (page 40)
Exercises: 7, 9, 11, 13 (page 49)

Problem 1. *Let F be a field and V be a vector space over F.*

(1) *Show that the scaler product of the zero element of F with any vector in V is the zero vector.*
(2) *Assume that $\dim_F V < \infty$. Let S be a finite subset which spans V, show that S contains a basis of V, namely, one can obtain a basis of V by deleting several elements from S.*
(3) *Assume that $\dim_F V = n$ and $S = \{\alpha_1, \ldots, \alpha_n\}$ is a set of n vectors. If S is linearly independent, show that S spans V and thus is a basis of V.*
(4) *Assume that $\dim_F V = n$ and $S = \{\alpha_1, \ldots, \alpha_n\}$ is a set of n vectors. If S spans V, show that S is linearly independent and thus is a basis of V.*

Problem 2. *Let F be a field and m, n be positive integers.*

(1) *Show that $\dim_F \mathrm{Mat}_{m\times n}(F) = mn$.*
(2) *For any $A \in \mathrm{M}_{n\times n}(F)$. Show that there is an integer N such that A satisfies a nontrivial polynomial equation*

$$A^N + c_{N-1}A^{N-1} + \cdots + c_1 A + c_0 I_n = 0.$$

(Hint: Consider the set $\{I_n, A, A^2, \ldots, A^N\} \subset M_{n\times n}(F)$ for a suitable integer N. Then apply Theorem 4 of page 44 and part (1).)

Problem 3. *In this problem, you are assumed to understand what a polynomial (of one variable/2-variables) is.*

(1) *Let $x(t), y(t)$ be quadratic polynomials with real coefficients (for example, $x(t) = t^2 + t + 1, y(t) = t^2 - 2t - 3$). Show that there is a quadratic polynomial $f(x, y)$ (i.e., $f(x, y)$ is of the form $c_0 + c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2$) such that $f(x(t), y(t))$ is identically zero.*
(2) *Let $x(t) = t^2 - 1, y(t) = t^3 - t$. Find a nonzero real polynomial $f(x, y)$ such that $f(x(t), y(t))$ is identically zero.*
(3) *Prove that every pair $x(t), y(t)$ of real polynomials satisfies some real polynomial relation $f(x, y) = 0$.*

(Hint: Use similar method as the last one).

Problem 4. *Let α be a real cubic root of 2 (i.e., $\alpha = \sqrt[3]{2}$).*

(1) *Show that the set $\{1, \alpha, \alpha^2\}$ are linearly independent over $\mathbb{Q}$, i.e., if $a, b, c \in \mathbb{Q}$ such that $a + b\alpha + c\alpha^2 = 0$, then $a = b = c = 0$.*
(2) *Show that the set $F = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathbb{Q}\}$ is a field.*

(Hint for (1): Proof by contradiction. Assume that $a, b, c \in \mathbb{Z}$ such that $a + b\alpha + c\alpha^2 = 0$ and divide $x^3 - 2$ by $cx^2 + bx + a$.)

Problem 5. *Set $\alpha = \sqrt[3]{2}$. Let $F = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$. We know that F is a field and has dimension 3 over $\mathbb{Q}$ from last problem. Thus F^n has dimension $3n$ as a $\mathbb{Q}$ vector space. Choose an ordered basis $\mathcal{B}$ of F^n (as a $\mathbb{Q}$-vector space). Compute $[v]_{\mathcal{B}}$ for $v = (v_1, \ldots, v_n) \in F^n$, where $v_i = a_i + b_i\alpha + c_i\alpha^2$.*

You **don't** have to do the next problem.

Problem 6. *Let V be a vector space over a field F. A subspace W of V is called proper if $W \neq V$. Show that if F is infinite, V is not the union of finitely many proper subspaces.*

(You can find a proof from this link. Please try to fill the details of the proof. You don't have to submit a solution of this problem.)