HOMEWORK 6

Due date: Nov 6, Monday of Week 11.

Exercises: 5, 11, 12, page 83-84
Exercises: 5, page 86
Exercises: 2, 10, 11, 12, page 95-96

Given a set X and a positive integer n, recall that the notation X^n denotes the Cartesian product $X \times X \times \cdots \times X$ (n-times) and an element of X^n is of the form $(x_1, \ldots, x_n)$ with $x_i \in X$. Let F be a field and let W be a vector space over F. Then W^n has a natural vector space structure: its addition and scaler multiplication are defined by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n);$$

$$c(x_1, \ldots, x_n) := (cx_1, \ldots, cx_n).$$

Here the notation $:=$ has the following meaning: the right side of this notation is the definition of the left side, or its left side is defined in terms of the right side. It should be clear that $\dim_F W^n = n \dim_F W$ if $\dim_F W$ is finite.

Problem 1. *Let V, W be vector spaces over F with $\dim_F V = n$. We don't assume that W has finite dimension. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a fixed (ordered) basis of V.*

(1) *Consider the linear transformation $\theta_{\mathcal{B}} : \mathrm{Hom}_F(V, W) \to W^n$ (the notation $\theta_{\mathcal{B}}$ means that this linear transformation depends on $\mathcal{B}$) defined by*

$$\theta_{\mathcal{B}}(T) = (T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)), \forall T \in \mathrm{Hom}_F(V, W).$$

Show that $\theta_{\mathcal{B}}$ is an isomorphism directly (without comparing dimensions).

(2) *Conclude that $\dim_F \mathrm{Hom}_F(V, W) = n \dim_F W$ if $\dim_F W$ is also finite.*

(Comment: Part (1) is just a restatement of Theorem 1, page 69. A special case of (2) is the isomorphism $\mathrm{Hom}_F(F^n, W) \cong W^n$. An even more special case is $\mathrm{Hom}_F(F, W) = W$.)

Problem 2. *Let V, W be two vector spaces over F and $T : V \to W$ be a linear transformation. Show that there is isomorphism*

$$\overline{T} : V/\ker(T) \to \mathrm{Im}(T)$$

defined by $\overline{T}(\alpha + \ker(T)) := T(\alpha)$.

The proof is given in class. Please repeat it here.

Problem 3. *Given $V_1, V_2, V_3 \in$ **Vect**$_F$ and $T_1 \in \mathrm{Hom}_F(V_1, V_2), T_2 \in \mathrm{Hom}_F(V_2, V_3)$. Show that*

(1) $\ker(T_1) \subset \ker(T_2 \circ T_1)$;
(2) $T_1(\ker(T_2 \circ T_1)) \subset \ker(T_2)$.
(3) *There is an injective map $\widetilde{T_1} : \ker(T_2 \circ T_1)/\ker(T_1) \to \ker(T_2)$ defined by $\widetilde{T_1}(\bar{x}) = T_1(x)$ for $x \in \ker(T_2 \circ T_1)$, where $\bar{x}$ represents the equivalence class of x, namely $\bar{x} = x + \ker(T_1)$. (You need to check that this map is well-defined, linear and injective. You can omit the "linear" part if you think it is easy.)*
(4) *Assume that $\dim_F(V_i) < \infty$ for $i = 1, 2, 3$. Show that $\dim_F \ker(T_2 \circ T_1) \leq \dim_F \ker(T_1) + \dim_F \ker(T_2)$.*
(5) *Given $A_2 \in \mathrm{Mat}_{m \times n}(F)$ and $A_1 \in \mathrm{Mat}_{n \times k}(F)$, show that*

$$\mathrm{rank}(A_1) + \mathrm{rank}(A_2) - n \leq \mathrm{rank}(A_2 A_1) \leq \min\{\mathrm{rank}(A_1), \mathrm{rank}(A_2)\}.$$

Hint: Consider the linear transformation $T_1 : F^k \to F^n, T_2 : F^n \to F^m$ defined by $T_i(X) = A_i X$, and use (4).

(6) *Given a matrix* $A \in \mathrm{Mat}_{m\times n}(F)$ *with* $\mathrm{rank}(F) = r$ *and a positive number* p *with* $p < r$. *Show that there does* ***not*** *exist matrices* $C \in \mathrm{Mat}_{m\times p}(F)$ *and* $R \in \mathrm{Mat}_{p\times n}(F)$ *such that* $A = CR$. *Hint: This is a direct corollary of the last part.*

Comment: The inequality in (5) is called Sylvester's rank inequality. Compare (6) with Problem 2 of HW 5. Do this problem step by step. It is not hard at all.

Problem 4. (1) *Show that* $\mathbb{C}^n$ *is isomorphic to* $\mathbb{R}^{2n}$ *as an* $\mathbb{R}$*-vector space. Namely, there exists an* $\mathbb{R}$*-linear isomorphism* $T : \mathbb{C}^n \to \mathbb{R}^{2n}$.

(2) *Given* $v = (z_1, \ldots, z_n) \in \mathbb{C}^n$ *with* $z_j = a_j + b_j\sqrt{-1} \in \mathbb{C}$ *with* $a_j, b_j \in \mathbb{R}$. *Compute* $T(v)$ *for the isomorphism* T *you choose in part (1).*

Problem 5. *Consider the* $\mathbb{R}$*-vector space* $V = \mathbb{C}$. *Then* $\dim_{\mathbb{R}}(V) = 2$. *Denote* $e_1 = -1 + 7i$ *and* $e_2 = 5i$, *where* $i = \sqrt{-1} \in \mathbb{C}$.

(1) *Show that* $\mathcal{B} = \{e_1, e_2\}$ *is a basis of* V *over* $\mathbb{R}$.

(2) *Let* $z = x + yi \in \mathbb{C}$ *with* $x, y \in \mathbb{R}$. *Consider the map* $f_z : V \to V$ *defined by* $f_z(t) = zt$. *Show that* f_z *is* $\mathbb{R}$*-linear.*

(3) *Let* $T(z) \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ *be the matrix of* f_z *with respect to the ordered basis* $\mathcal{B}$. *Show that*

$$T(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}.$$

Note that the matrix $T(z)$ is the one given in Problem 5 of page 86 of the textbook. By Problem 5 of Page 86, the map $z \mapsto T(z)$ is injective, and satisfies $T(z_1z_2) = T(z_1)T(z_2)$. These properties can be proved without explicit calculation, just using the fact that $T(z)$ is the matrix of the linear transformation f_z. The next problem is a "higher dimensional" version of Problem 5 of page 86.

Problem 6. *Consider the* $\mathbb{R}$*-vector space* $V = \mathbb{C}^2$. *Then* $\dim_{\mathbb{R}}(V) = 4$.

(1) *Find an ordered basis* $\mathcal{B}$ *of* V *over* $\mathbb{R}$.

(2) *For a matrix* $A \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$, *consider the map* $T_A : V \to V$ *defined by* $T_A(X) = AX$ *for* $X \in \mathbb{C}^2$, *which is* $\mathbb{R}$*-linear. Denote* $R(A) = [T_A]_{\mathcal{B}}$, *namely* $R(A) \in \mathrm{Mat}_{4\times 4}(\mathbb{R})$ *is the matrix of* T_A *with respect to* $\mathcal{B}$ *when* V *is viewed as a* $\mathbb{R}$*-vector space. Show that the map* $A \mapsto R(A)$ *from* $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ *to* $\mathrm{Mat}_{4\times 4}(\mathbb{R})$ *satisfies* $R(A)R(B) = R(AB)$.

(3) *Compute* $R(I_2)$, *where* $I_2 \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$ *is the identity matrix.*

(4) *Show that if* A *is invertible, then* $R(A)$ *invertible.*

(5) *Let* $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. *Compute* $R(A)$. *(It depends on the ordered basis you chose in part (1)).*

The following problem is similar to the above problem in a slightly different situation. Part of it has been covered in class. You don't have to submit a solution because it is not a very specific problem. But you are advised to do it. It is possible that we will have a problem like this in our next exam.

Problem 7. *Denote* $\alpha = \sqrt[3]{2}$. *Consider* $F = \{a + b\alpha + c\alpha^2 | a, b, c \in \mathbb{Q}\}$. *We know that* F *is a field and it is also a vector space over* $\mathbb{Q}$ *of dimension 3 from previous HW. We view it as a* $\mathbb{Q}$*-vector space.*

(1) *Given* $x = a + b\alpha + c\alpha^2 \in F$ *and the linear map* $T_x : F \to F$ *given by* $T_x(y) = xy$. *It is not hard to see* T_x *is well-defined and* $\mathbb{Q}$*-linear. Suppose that* $x \neq 0$. *Show that* T_x *is injective and conclude that there exists a* $y \in F$ *such that* $xy = 1$.

(2) *Fix an ordered basis* $\mathcal{B}$ *of* F *(as a* $\mathbb{Q}$ *vector space) and compute the matrix* $[T_x]_{\mathcal{B}}$ *of* T_x *with respect to the basis you chose.*

(3) *Show that* $[T_x]_{\mathcal{B}}$ *is invertible (this is similar to part (4) of Problem 6.)*

(4) *Do a higher dimensional analogue of this. For example, given a matrix* $A \in \mathrm{Mat}_{2\times 2}(F)$, *and consider the linear map* $T_A : F^2 \to F^2$. *View* F^2 *as a* $\mathbb{Q}$*-vector space and choose a basis* $\mathcal{B}'$ *of* F^2 *over* $\mathbb{Q}$. *Then compute* $[T_A]_{\mathcal{B}'} \in \mathrm{Mat}_{6\times 6}(\mathbb{Q})$ *explicitly in terms of entries of* A. *Show that if* A *is invertible as an element of* $\mathrm{Mat}_{2\times 2}(F)$ *then* $[T_A]_{\mathcal{B}}$ *is invertible as an element of* $\mathrm{Mat}_{6\times 6}(\mathbb{Q})$.