HOMEWORK 8

Due date: Nov 20, Monday of Week 13
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Exercises: 1, 2, 3, 4, 5, 6, page 126-127

Let $\mathbb{F}_p$ be the field with p-elements, where p is a prime number. Recall that this field is constructed using equivalence classes. A different way to write this field is $\mathbb{Z}/p$.

Let F be a fixed field. Let $V_n = V_n(F)$ be the F-vector space of F-polynomials of degree $\leq n$. Then $\dim_F V_n = n+1$ and thus $\dim_F V_n^* = n+1$. Given $t \in F$, we have defined $L_t \in V_n^*$ by $L_t(f) = f(t)$ for $f \in V_n$. Lagrange interpolation says that if $t_0, \dots, t_n$ are distinct points in F, then $\{L_{t_i} : 0 \leq i \leq n\}$ is a basis of V_n^*.

Problem 1. *Let $F = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ be the field of 5 elements. Consider $V_3^*(F)$ which has dimension 4 and thus $L_0, L_1, L_2, L_3, L_4 \in V_3^*$ are linearly dependent. Write L_4 as a linear combination of L_0, L_1, L_2, L_3.*

Problem 2. *Let S be any subset of F. Show that the subset $\{L_s : s \in S\} \subset \mathrm{Hom}_F(F[x], F)$ is linearly independent where L_s is viewed as a linear function on $F[x]$ defined by $L_s(f) = f(s), f \in F[x]$.*

The formal power series algebra $F[[x]]$ is written as F^∞ in the book. We also used the notation $F^{\mathbb{N}}$ to denote $F[[x]]$.

Problem 3. *Given a formal power series $f = (f_0, f_1, \dots, f_n, \dots,) \in F^{\mathbb{N}}$, we consider the map $\phi_f : F[x] \to F$ defined by*

$$\phi_f(x^i) = f_i$$

or

$$\phi_f(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 f_0 + a_1 f_i + \cdots + a_n f_n.$$

(1) *Show that ϕ_f is linear and thus $\phi_f \in \mathrm{Hom}_F(F[x], F)$.*
(2) *Show that the map $\phi : F[[x]] \to \mathrm{Hom}_F(F[x], F)$ is linear as F-vector spaces.*
(3) *Show that ϕ is an isomorphism by explicitly constructing an inverse of ϕ.*
(4) *Given an element $t \in F$. By part (3), we know that L_t must be of the form ϕ_f for some $f \in F[[x]]$. Describe f in terms of t.*

(Comment: Even we cannot compare dimensions because both $F[x]$ and $F[[x]]$ are infinite dimensional as F-vector spaces, it should be clear that $F[[x]]$ is strictly larger than $F[x]$. Actually one can show that $F[x]$ and $F[[x]]$ are not isomorphic as F-vector spaces (You might search a proof of this online. But we won't show it in this course.) This problem shows that the dual of $F[x]$ is strictly larger than $F[x]$, which never happens in the finite dimension case.)

(Comment: In HW6, Problem 1, we know that $\mathrm{Hom}(V, W) \cong W^n$ if $\dim V = n$. From this problem we know that $\mathrm{Hom}(F[x], F) \cong F[[x]]$. Both isomorphisms is a special case of something you might learn in the future.)

You can do the above 3 problems after Wednesday's class.

Problem 4. (1) *Find a nonzero polynomial $f \in \mathbb{F}_p[x]$ such that $f(a) = 0$ for any $a \in \mathbb{F}_p$.*
(2) *Let $f \in \mathbb{F}_p[x]$ be a nonzero polynomial such that $f(a) = 0$ for any $a \in \mathbb{F}_p$. Show that $\deg(f) \geq p$.*
(3) *Consider the set $I = \{f \in \mathbb{F}_p[x] : f(a) = 0, \forall a \in \mathbb{F}_p\}$. Show that I is ideal of $F[x]$.*

Hint: (2) is a consequence of Lagrange interpolation. Of course, you can also compare the number of roots of a polynomial and the degree of the given polynomial.

Problem 5. *Let* $A \in \mathrm{Mat}_{n\times n}(F)$ *be a fixed non-zero polynomial. Consider the set*

$$I = \{f \in F[x] : f(A) = 0\}.$$

Show that I *is a (nonzero) ideal. Suppose that* d *is the nonzero monic polynomial such that* $I = dF[x]$. *Show that* $\deg(d) \leq n^2$.

You can do the above two problems after Friday's class.