HOMEWORK 1

Due date: Monday of Week 2
Exercises: 1, 2, 5, 9, 10, page 213
Exercises: 3, 5, pages 218-219
Exercises: 2, 5, 6, 13, 14, 15.

In the following F is a general field. If it is necessary, you can assume that the characteristic of F is zero.

This's a problem from last final exam. Do it again on your own.

Problem 1. *Consider the following two matrices in* $\mathrm{Mat}_{3\times 3}(\mathbb{R})$

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -3 & -2 & -3 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 2 \\ -3 & -6 & -3 \\ 2 & 2 & -1 \end{bmatrix}.$$

Determine whethere A, B can be simultaneously diagonalizable. If so, find a matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

Problem 2. *Given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$. Suppose that $I - A$ is invertible, where $I \in \mathrm{Mat}_{n\times n}(F)$ is the identity matrix. Show that $(I - A)^{-1}$ is a polynomial of A, i.e., there exists a polynomial $f \in F[x]$ such that $(I - A)^{-1} = f(A)$.*

Hint: If A is nilpotent, it should be easy to find such f. For the general A, consider the characteristic polynomial $\chi_A(x) = \det(xI - A)$. The assumption shows that $\chi_A(1) \neq 0$. Consider the long division $\chi_A(x) = (x - 1)p + r$ and use Cayley-Hamilton. You can use this problem to do Exercise 9 of page 213. It is easy if you could guess its answer. But this problem will tell you how to guess the answer. See also Exercise 5, page 213.

Let V be a finite dimensional vector space over a field F and let $T \in \mathrm{End}_F(V)$. Let W be a T-invariant subspace of V with $W \neq 0, W \neq V$. We can consider the following question. Is there always a T-invariant subspace W' of V such that $V = W \oplus W'$. The following is an example.

Problem 3. *Let V be a vector space over F with $\dim_F V = 2$. Let $\mathcal{B} = \mathrm{Span}\{\alpha_1, \alpha_2\}$ be a basis of V. Consider*

$$T : V \to V$$

defined by $T(\alpha_1) = \alpha_1, T(\alpha_2) = a\alpha_1 + \alpha_2$ for some $a \in F$. Let $W = \mathrm{Span}\{\alpha_1\}$. Then W is T invariant. Determine whether there exists a $W' \subset V$ such that W' is T-invariant and $V = W \oplus W'$.

This problem is similar to Exercise 2, page 218. We will go back to this problem in Section 7.5.

Problem 4. *Let $\rho : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$ be a map such that $\rho(z_1z_2) = \rho(z_1)\rho(z_2), \forall z_1, z_2 \in \mathbb{C}^\times$.*

(1) *Construct such a ρ such that $\rho(z)$ is not diagonalizable for any $|z| \neq 1$;*
(2) *Suppose that there is an element z_0 such that $\rho(z_0)$ has two distinct eigenvalues. Show that there exists a matrix $P \in \mathrm{GL}_2(\mathbb{C})$ such that*

$$P^{-1}\rho(z)P$$

is diagonal for any $z \in \mathbb{C}^\times$.

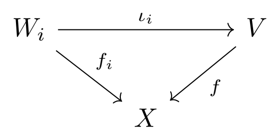
Hint for (1): think about what kind matrices in $\mathrm{GL}_2(\mathbb{C})$ are not diagonalizable.

Problem 5. *Consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ with p-elements for a prime p. Consider $\mathbb{F}_p^\times = \{x \in \mathbb{F}_p : x \neq 0\}$. It is known that for any element $x \in \mathbb{F}_p^\times$, we have $x^{p-1} = 1$. This is called Fermat's little theorem. We will show this later (or you can try to prove this on your own). Let $\rho : \mathbb{F}_p^\times \to \mathrm{GL}_n(\mathbb{C})$ be a map such that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$ for any $x_1, x_2 \in \mathbb{F}_p^\times$.*

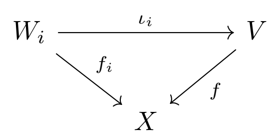
(1) *Show that* $\rho(1) = I_n$.
(2) *Show that there exists an element* $P \in \mathrm{GL}_n(\mathbb{C})$ *such that* $P\rho(x)P^{-1}$ *is diagonal for any* $x \in \mathbb{F}_p^{\times}$.

Problem 6. *Let* $W_i, 1 \leq i \leq k$ *be subspaces of* V. *Let* $\iota_i : W_i \to W$ *be the linear map defined by* $\iota_i(\alpha_i) = \alpha_i$. *This makes sense because* W_i *is a subspace of* V.

(1) *Suppose*
$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$
Given any vector space X *and any linear map* $f_i : W_i \to X$, *show that there is a unique linear map* $f : V \to X$ *such that* $f_i = f \circ \iota_i$ *for each* i *with* $1 \leq i \leq k$. *In other words, there exists a commutative diagram*



(2) *Given any vector space* X *and any linear map* $f_i : W_i \to X$, *suppose there is a unique linear map* $f : V \to X$ *such that* $f_i = f \circ \iota_i$ *for each* i *with* $1 \leq i \leq k$. *In other words, there exists a commutative diagram*



show that
$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

Given a vector space V and two subspaces W_1, W_2 such that $V = W_1 \oplus W_2$. Let $E_i : V \to W_i \subset V$ be the corresponding projection. Now the question is: how to write E_i explicitly using matrix. Consider the following example.

Problem 7. *Let* $V = F^3$. *The elements of* V *are viewed as column vectors. Let* $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t$, *and let* $\mathcal{B} = [\epsilon_1, \epsilon_2, \epsilon_3]$ *be the standard basis of* V. *Let* $\alpha_1 = \epsilon_1, \alpha_2 = \epsilon_2$ *and* $\alpha_3 = 19\epsilon_1 + 5\epsilon_2 + \epsilon_3$. *Let* $W_1 = \mathrm{Span}\{\alpha_1, \alpha_2\}, W_2 = \mathrm{Span}\{\alpha_3\}$. *Then it is easy to see that* $V = W_1 \oplus W_2$. *Let* E_i *be the corresponding projection. Compute the matrices*
$$[E_i]_{\mathcal{B}}$$
for $i = 1, 2$.

Suppose that F is a field of characteristic zero. The next problem is similar to Exercise 2, page 225.

Problem 8. *Consider the matrix*
$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}.$$

(1) *Show that* A *is not diagonalizable.*
(2) *Find the Jordan decomposition of* A, *namely, find a diagonalizable matrix* D *and a nilpotent matrix* N *such that* $A = D + N$ *and* $DN = ND$.

You don't have to submit solutions of the rest problems. But it is worth to think about them. These problems should be with you when you read the book. But I don't have time to address these questions in classes. Again, try to ask yourself reasonable questions when you read math books. These problems are related to Exercise 4, page 225. Given a $T \in \mathrm{End}_F(V)$.

Problem 9. *In the primary decomposition theorem, if* $\mu_T = p_1^{r_1} \dots p_k^{r_k}$*, we have*

$$V = W_1 \oplus \dots \oplus W_k,$$

with $W_i = \ker(p_i(T)^{r_i})$*. What can you say about* $\dim W_i$*? Show that* $W_i \neq 0$ *at least.*

Problem 10. *Let* $f \in I(T) = \{g \in F[x] : g(T) = 0\}$ *be a nonzero polynomial and let*

$$f = p_1^{s_1} \dots p_t^{s_t}$$

be the prime decomposition of f *with distinct irreducible polynomials* $p_1, \dots, p_k$ *and* $s_i \geq 0$*. Let* $W_i' = \ker(p_i(T)^{s_i})$*. Show that*

1. $V = W_1' \oplus W_2' \oplus \dots \oplus W_t'$;
2. *each* W_i' *is* T*-invariant.*
3. *If* $p_j \nmid \mu_T$*, show that* $W_j' = \{0\}$*.*
4. *If* $p_i | \mu_T$ *and* $p_i | f$*, show that* $W_i = W_i'$*, namely,* $\ker(p_i(T)^{r_i}) = \ker(p_i(T)^{s_i})$*.*
5. *From the last two parts, apparently, we cannot expect* $p_i^{s_i}$ *is the minimal polynomial of* $T|_{W_i'}$*.*

Problem 11. *Do we know that* μ_T *and* χ_T *have exactly the same prime factors? Is it possible that there exists a prime polynomial such that* $p | \chi_T$ *but* $p \nmid \mu_T$*?*

The answer is no. See section 7.2. But it is a good exercise to think about this by yourself. We proved that μ_T and χ_T have the same roots. But it is possible that over a field, a prime factor p does not have any root. So if there exists a prime factor $p | \chi_T$ but $p \nmid \mu_T$, it does not contradict to what we know so far. But why cannot this happen?

Now suppose that

$$\chi_T = p_1^{d_1} \dots p_k^{d_k}.$$

Now let $W_i' = \ker(p_i(T)^{d_i})$ for $1 \leq i \leq k$. The above problems show that $W_i' = W_i$ and the primary decomposition obtained using χ_T and the primary decomposition obtained using $f = \mu_T$ are exactly the same.

Now think about the following question:

Problem 12. *Why do we use* μ_T *rather than* χ_T *in the statement of the primary decomposition theorem even it gives the same decomposition when we replace* μ_T *by* χ_T*?*

To help you to understand the above decompositions, try to work out the following example.

Problem 13. *Let* $V = F^4$ *and* $T : V \to V$ *is represented by*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have $\mu_T = x^2(x-1)$ *and* $\chi_T = x^2(x-1)^2$*. Also consider the polynomial* $g = x^2(x-1)(x^2+1) \in I(T)$*. Compute* $W_1 = \ker(T^2), W_2 = \ker(T-1)$*; and* $W_1' = \ker(T^2)$ *and* $W_2' = \ker(T-I)^2$*; and* $\ker(T^2+I)$*. Here we assume that the characteristic of* F *is not* 2*.*