HOMEWORK 5

Due date: Monday of Week 6

Exercises: 1, 2, 3, 6, 11, 14, pages 288-290
Exercises: 1, 2, 4, 5, 6, 7, 10, 12, pages 298-299.

Assume that $F = \mathbb{R}$ or $\mathbb{C}$.

Problem 1. *In Theorem 7, page 293 of the textbook, we only defined adjoint for linear operators $T \in \mathrm{End}(V)$. Try to generalize this concept to general linear maps, namely, try to define the adjoint T^* for $T \in \mathrm{Hom}(V, W)$, where V, W are two (possibly different) inner product spaces over F. Moreover, show that the adjoint you defined above indeed exists.*

Problem 2. *Let $A \in \mathrm{Mat}_{m\times n}(F)$. We consider the linear operator $T_A : F^n \to F^m$ by $T_A(\alpha) := A\alpha$. Here F^n and F^m are viewed as inner product space with respect to the standard inner product defined on them. Show that the adjoint of T_A is given by T_{A^*}, where $A^* = \overline{A^t}$ and the adjoint of T_A is defined in the last problem.*

We consider the column vector space $\mathbb{R}^n$ over $\mathbb{R}$. Let $(\,|\,)$ be the standard inner product on $\mathbb{R}^n$. Recall that $(x|y) = y^t x$, where y^t denotes the transpose of y. Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, we define the linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(x) = Ax$. Denote $\mathrm{Ker}(A) = \{x \in \mathbb{R}^n : Ax = 0\} = \mathrm{Ker}(T_A)$. Let $\mathrm{Row}(A)$ denote the space spanned by rows of A.

Problem 3. *Given $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Show that*

(1) $\mathrm{Ker}(A) = \mathrm{Ker}(A^t A)$;
(2) $\mathrm{rank}(A) = \mathrm{rank}(A^t A)$;
(3) $\mathrm{Row}(A) = \mathrm{Row}(A^t A)$.

Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional inner product space over F and let W be a subspace of V. We then have the orthogonal decomposition $V = W \bigoplus W^\perp$. Let $Proj_W : V \to V$ denotes the projection from V to W corresponding to this decomposition, namely, $Proj_W(\alpha, \beta) = \alpha$ for $\alpha \in W, \beta \in W^\perp$.

Problem 4. *Let $V = \mathbb{R}^n$ endowed with the standard inner product. Let $W \subset V$ be a subspace of V of dimension m. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_m\}$ be a basis of W and consider the matrix*

$$M_{\mathcal{B}} = [\alpha_1, \dots, \alpha_m] \in \mathrm{Mat}_{n\times m}(\mathbb{R}).$$

Here each α_i is a column vector.

(1) *Show that $M_{\mathcal{B}}^t M_{\mathcal{B}} \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ is invertible.*
(2) *Consider the matrix $P_{\mathcal{B}} = M_{\mathcal{B}}(M_{\mathcal{B}}^t M_{\mathcal{B}})^{-1} M_{\mathcal{B}}^t \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that $P_{\mathcal{B}}$ is independent on the choice of $\mathcal{B}$ and thus it only depends on the space W.*
(3) *For any $\alpha \in \mathbb{R}^n$, show that $P_{\mathcal{B}}\alpha \in W$. (The notation $P_{\mathcal{B}}\alpha$ denotes the matrix product of $P_{\mathcal{B}}$ with α).*
(4) *Show that the map $E : \mathbb{R}^n \to \mathbb{R}^n$ defined by $E(\alpha) = P_{\mathcal{B}}\alpha$ is the same as the projection map $Proj_W$. (Hint: one way to do this is by choosing a good basis of W using the last part. Then compute the matrix $P_{\mathcal{B}}$.)*

Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $\beta \in \mathbb{R}^m$, we consider the linear system

$$Ax = \beta. \tag{0.1}$$

The above equation does not always have a solution. If the above equation has no solution, we can consider the following approximating solution, which is called *least square solution.* A vector $\hat{x} \in \mathbb{R}^n$ is called a least square solution of (0.1) if

$$||\beta - A\hat{x}|| \le ||\beta - Ax||, \forall x \in \mathbb{R}^n.$$

It is clear that if $x \in \mathbb{R}^n$ is a solution of (0.1), then it is also a least square solution.

Problem 5. *Given* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *and* $\beta \in \mathbb{R}^m$*. Consider*

$$\mathrm{Im}(A) = \{y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n\} \subset \mathbb{R}^m,$$

and

$$\mathrm{Ker}(A^t) = \left\{y \in \mathbb{R}^m : A^t y = 0\right\} \subset \mathbb{R}^m.$$

(1) *Show that* $\mathrm{Im}(A)^\perp = \mathrm{Ker}(A^t)$*. Here* $\perp$ *is relative to the standard inner product on* $\mathbb{R}^m$*.*
(2) *Show that* $\hat{x} \in \mathbb{R}^n$ *is a least square solution of* (0.1) *if and only if* $\beta - A\hat{x} \in \mathrm{Im}(A)^\perp$ *if and only if* $A^t A\hat{x} = A^t\beta$*.*
(3) *Show that* (0.1) *always has a least square solution.*
(4) *Give a condition such that* (0.1) *has a unique least square solution.*

Problem 6. *Let* $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ *and* $\beta = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$*. Find a least square solution of the equation*

$$Ax = \beta.$$

Problems 3-6 were stated for the field $\mathbb{R}$. Try to consider the analogues for inner product spaces over $\mathbb{C}$. For example, in Problem 3, if we replace $\mathbb{R}$ by $\mathbb{C}$ and replace A^t by A^*, then show the same assertions hold. In Problem 4, what is the matrix $P_{\mathcal{B}}$ if the field is $\mathbb{C}$? Formulate and solve the least square solution problem over $\mathbb{C}$ as in Problem 5.