HOMEWORK 7

Due date: Monday of Week 8

You might assume the field is $\mathbb{C}$ in the exercises of the book if it helps. Properties of normal operators over $\mathbb{R}$ will be proved later.
Exercises: 6, 7, 8, 9, pages 324-325.
Exercises: 4, 6, 9, 10, 11, 13, 14, 15, page 331-332
Exercises: 2, 3, 6, pages 347.

Let V be a finite dimensional vector space over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. Let $T \in \operatorname{End}(V)$ be a diagonalizable normal operator with spectral decomposition

$$T = c_1 E_1 + \cdots + c_k E_k.$$

Recall that each E_i is a polynomial of T. In fact, we have

$$E_j = \prod_{i \neq j} \frac{T - c_i I}{c_j - c_i}$$

see Corollary, page 336. Let $S = \{c_1, \ldots, c_k\}$ and $f : S \to F$ be a function. Then we define

$$f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k.$$

In general, the expression $f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k$ need not to be the spectral decomposition of $f(T)$, because it is possible that $f(c_i) = f(c_j)$ even $i \neq j$. Consider the simple example when $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $f(x) = x^2$ for $x \in \{-1, 1\}$. In this case $f(T) = E_1 + E_2$ is not the spectral decomposition of $f(T)$. If f is injective, then $f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k$ is the spectral decomposition of $f(T)$. (Check this!) In particular, E_i is also a polynomial of $f(T)$. This can be checked directly as in the following problem.

Problem 1. *Let $f : S \to F$ be an injective map. Show that*

$$E_j = \prod_{i \neq j} \frac{f(T) - f(c_i)I}{f(c_j) - f(c_i)}$$

by a direct computation.

Problem 2. *Let $f : S \to F$ be an injective map. Let $U \in \operatorname{End}(V)$ be another linear operator. Show that U commutes with T if and only if U commutes with $f(T)$.*

Problem 3. *Let V be a finite dimensional inner product space over F with $F = \mathbb{R}$ or $\mathbb{C}$. Let T be a nonnegative operator on V and N be the unique non-negative operator on V such that $T = N^2$. Let $U \in \operatorname{End}(V)$. Show that U commutes with N if and only if U commutes with T.*

This is a special case of Problem 2.

Exercise 4 of page 347 collects some equivalent definitions of normal operators. Many of them were proved in class and in last HW. In the next problem, show the equivalence of (a) and (i).

Problem 4. *Let V be a finite dimensional inner product space over F with $F = \mathbb{R}$ or $\mathbb{C}$. Let $T \in \operatorname{End}(V)$. Show that T is normal if and only $UN = NU$, where N is non-negative and U is unitary such that $T = UN$, namely, $T = UN$ is the polar decomposition of T.*

This is a consequence of Problem 3.

Problem 5. *For any $A \in \operatorname{Mat}_{n \times n}(\mathbb{R})$, show that there exists $P_1, P_2 \in \mathrm{O}_n(\mathbb{R})$ and a diagonal matrix D such that $A = P_1 D P_2$.*

This is called the singular decomposition of A. We talked this in class. Repeat it here.

Problem 6. *Consider the matrix*

$$A = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}).$$

Find a singular decomposition of A.

Problem 7. *Denote* $\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : AA^t = I_n\}$ *and* $\mathrm{SO}_n(\mathbb{R}) = \{g \in \mathrm{O}_n(\mathbb{R}) : \det(g) = 1\}$. *We assume that n is even, namely $n = 2m$ for a positive integer m. Given $A \in \mathrm{O}_n(\mathbb{R}) \smallsetminus \mathrm{SO}_n(\mathbb{R})$ (this means $A \in \mathrm{O}_n(\mathbb{R})$ but $A \notin \mathrm{SO}_n(\mathbb{R})$).*

(1) *If $\lambda \in \mathbb{C}$ is an eigenvalue of A, show that $\lambda\overline{\lambda} = 1$ and $\overline{\lambda}$ is also an eigenvalue of A.*
(2) *Show that -1 must be an eigenvalue of A and $\dim_{\mathbb{R}} E_A(-1)$ is odd. Here $E_A(-1) = \{\alpha \in \mathbb{R}^n : A\alpha = -\alpha\}$ is the eigenspace corresponding to -1.*
(3) *Let $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a matrix such that $AB = BA$. Show that $E_A(-1)$ is invariant under the left multiplication by B and B must have a real eigenvalue.*
(4) *Consider the matrix $P = \begin{bmatrix} & -I_m \\ I_m & \end{bmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. If $Q \in \mathrm{O}_n(\mathbb{R})$ such that $QP = PQ$, show that $\det(Q) = 1$.*

Hint for (2): you can use the fact that $\dim_{\mathbb{C}} E_A(-1)_{\mathbb{C}} = \dim_{\mathbb{R}} E_A(-1)$, where $E_A(-1)_{\mathbb{C}} = \{\alpha \in \mathbb{C}^n : A\alpha = -\alpha\}$. This fact was proved in the solution of Ex section 7.2. You don't have to prove this fact again.