HOMEWORK 9

Due date: Monday of Week 10

Exercises: 4, 5, 7, 10, 12, 13, pages 378-379

Recall the definition of the group $O(p,q)$. Let $p, q \geq 0$ be two integers and set $n = p+q$. Let $V = \mathbb{R}^n$ and $f_{p,q} : V \times V \to \mathbb{R}$ be the bilinear form defined by

$$f_{p,q}(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_{p+j} y_{p+j}, \tag{0.1}$$

for $x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \in V$. If we write

$$s_{p,q} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1),$$

where there are p 1 in the diagonal and q -1 in the diagonal. Then

$$f(x,y) = y^t s_{p,q} x.$$

Consider the group

$$O(p,q) = \{g \in \mathrm{GL}_n(\mathbb{R}) : f(gx, gy) = f(x,y), \forall x, y \in V\}. \tag{0.2}$$

If $q = 0$ and $p = n$, we often write $O(n,0)$ as $O(n)$, which is just the orthogonal group defined in Chapter 8. The group $O(3,1)$ is called the Lorentz group, which is used in special relativity.

Problem 1. (1) *Show that* $O(p,q) = \{g \in \mathrm{GL}_n(\mathbb{R}) : g^t s_{p,q} g = s_{p,q}\}$.

In this problem, we show that $O(2)$ and $O(1,1)$ are different.

Problem 2. (1) *For any* $a \in \mathbb{R}^\times$*, show that*

$$A(a) := \begin{pmatrix} \frac{a+a^{-1}}{2} & \frac{a-a^{-1}}{2} \\ \frac{a-a^{-1}}{2} & \frac{a+a^{-1}}{2} \end{pmatrix} \in O(1,1).$$

Moreover, the map $A : \mathbb{R}^\times \to O(1,1)$ *satisfies* $A(ab) = A(a)A(b)$.

(2) *For any*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2),$$

show that each entry a, b, c, d *is bounded.*

A better way to realize the group $O(1,1)$ is to use the bilinear form $f' : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f'(x,y) = y^t \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} x.$$

Consider the group $G = \left\{g \in \mathrm{GL}_2(\mathbb{R}) : f'(gx, gy) = f'(x,y), \forall x, y \in \mathbb{R}^2\right\}$.

Problem 3. *Construct a bijective map* $\phi : G \to O(1,1)$ *such that* $\phi(gh) = \phi(g)\phi(h)$ *for any* $g, h \in G$.

Problem 4. *Describe all elements of the above group* G.

Reflection. Recall the formula of reflection on $\mathbb{R}^3$ endowed with the standard inner product. For a nonzero vector $v \in \mathbb{R}^3$, the reflection r_v about the plane P_v orthogonal to v is given by

$$r_v : \mathbb{R}^3 \to \mathbb{R}^3$$
$$r_v(x) = x - 2\frac{(v|x)}{||v||^2} v.$$

We have $r_v \in O(3)$. It is a fact that any element $g \in O(3)$ is a product of certain reflections. The following is a generalization.

Let F be a general field with characteristic zero, and let V be a finite dimensional vector space over F. Let $B : V \times V \to F$ be a non-degenerate symmetric bilinear form. We can define the orthogonal group

$$O(V, B) = \{g \in \mathrm{GL}(V) : B(gx, gy) = B(x, y), \forall x, y \in V\}.$$

Let $q : V \to F$ be the map defined by $q(v) = B(v, v)$. It is not hard to recover B from q as we did in Section 8.1 in some special cases. (Try this!) The pair (V, q) is usually called a quadratic space. For $v \in V$ with $q(v) \neq 0$, we define $r_v : V \to V$ by

$$r_v(x) = x - \frac{2B(x, v)}{q(v)} v.$$

Then one can show that $r_v \in O(V, B)$.

Problem 5. *Let $V = \mathbb{R}^n, B = f_{p,q}$ as defined in (0.1), show that r_v defined above is in $O(V, B)$, which is just $O(p, q)$ defined in (0.2).*

It is helpful to keep in mind the following

Theorem 0.1 (Cartan-Dieudonné)**.** *Any element $g \in O(V, B)$ is a product of a finite number of reflections.*

We won't prove the above theorem. For a proof, see Pete Clark's notes on quadratic forms.

We showed in class that product of two reflections is a rotation in $O(3, \mathbb{R})$. According the above theorem, any rotation in $\mathbb{R}^3$ is a product of two reflections. Try to check this directly.

Problem 6. *Consider the plane curve C given by*

$$x^2 + xy + y^2 = 2.$$

This means that $C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + xy + y^2 = 2\right\}$. Find a rotation $\rho \in \mathrm{O}_2(\mathbb{R})$ such that if we do the substitution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \rho \begin{bmatrix} x \\ y \end{bmatrix},$$

the equation of the curve becomes of the form

$$ax'^2 + by'^2 = c.$$

Determine the shape of C.