HOMEWORK 11

Due date:

Exercises: 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 9.12, 9.18, page 506-509 of Artin's book.
Also try 9.14 and 9.15. But you don't have to submit your work on these two problems.

Problem 1. *Let F be a field and $f \in F[x]$ be a separable polynomial of degree n. Show that f is irreducible iff G_f acts transitively on the roots of f.*

Note that G_f acts transitively on the roots of f means that G_f is a transitive subgroup of S_n.
Let F be a field, $f \in F[x]$ be a separable polynomial of degree 4 with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in an extension K. Consider

$$\begin{aligned} \alpha &= \alpha_1\alpha_2 + \alpha_3\alpha_4, \\ \beta &= \alpha_1\alpha_3 + \alpha_2\alpha_4, \\ \gamma &= \alpha_1\alpha_4 + \alpha_2\alpha_3. \end{aligned}$$

Let $R_f = (x-\alpha)(x-\beta)(x-\gamma)$, which is called the resolvent cubic of f.

Problem 2. *Let $f = x^4 + bx^3 + cx^2 + dx + e \in F[x]$ and let R_f be its resolvent cubic. Show that* $\mathrm{disc}(f) = \mathrm{disc}(R_f)$.

Hint: Use definitions.

Problem 3. *If $f = x^4+bx^3+cx^2+dx+e \in F[x]$, show that $R_f = x^3-cx^2+(bd-4e)x-b^2e+4ce-d^2$.*

Recall that a group G is called solvable if there exists a normal series

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$$

such that G_i/G_{i+1} is abelian for each i.

Problem 4. *Let G be a finite group. Show that G is solvable iff there exists a normal series*

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$$

such that G_i/G_{i+1} is cyclic for each i.

Let p be a prime integer. Recall that a finite group is called solvable if $|G| = p^e$ for some positive integer e.

Problem 5. *Show that any p-group is solvable.*

Hint: This is essentially Proposition 7.3.1, page 197 of Artin's book.
A famous theorem of Burnside says that if $|G| = p^a q^b$ for p, q prime and $a, b \in \mathbb{N}$, then G is solvable. Its proof is much harder.

Problem 6. *Let F be a field and let B_n be the upper triangular subgroup of $\mathrm{GL}_n(F)$. Show that B_n is solvable.*

Many matrices groups, like $\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F) (n \geq 3), \mathrm{Sp}_{2n}(F)$ are not solvable. See Theorem 9.8.4, page 282 of Artin's book. As an example, let $G = \mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$, try to compute the derived normal series $G^{(k)}$, where $G^{(1)} = [G,G]$ and $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$ for $k \geq 2$.
Given a group G, define $D_1G = [G,G] = G^{(1)}$, $D_2G = [G, D^1G], \ldots, D_kG = [G, D_{k-1}G]$. Then we have the normal series

$$D_kG \trianglelefteq D_{k-1}G \trianglelefteq \cdots \trianglelefteq D_1G \trianglelefteq G.$$

This series is called the lower central series of G. Notice that $G^{(k)} \subsetneq D_kG$ in general. A group G is called **nilpotent** if $D_kG = \{1\}$. Notice that if G is nilpotent, it must be solvable. The converse is false.

Problem 7. *Let F be a field and let B_n be the upper triangular subgroup of $\mathrm{GL}_n(F)$. Let $U_n \subset B_n$ be the subgroup with elements 1 in the diagonal. Show that B_n is not nilpotent but U_n is nilpotent.*

1. Discriminant of a special polynomial

Given $f = \prod(x - \alpha_i)$. Recall that $\mathrm{disc}(f) = \prod_{i \neq j}(\alpha_i - \alpha_j)^2$. Assume that K is a field of characteristic zero.

Problem 8. *Suppose $L = K(\beta)$ for some β and let $f := \mu_\beta$ be the minimal polynomial of β over K. Show that*

$$\mathrm{disc}(f) = (-1)^{\frac{m(m-1)}{2}} \mathrm{Nm}_{L/K}(f'(\beta)).$$

Here $m = \deg(f)$.

You might use the Norm formula in Problem 8, HW9.

Assume characteristic of K is zero. Consider the polynomial $f = x^n + ax + b \in K[x]$. We assume that f is irreducible. By last problem, we have

$$\mathrm{disc}(f) = \mathrm{Nm}_{L/K}(f'(\beta)),$$

where β is a root of f and $L = K(\beta)$. Denote $\gamma = f'(\beta) = n\beta^{n-1} + a$. To get $\mathrm{Nm}_{K(\beta)/K}(\gamma)$, it is better to find its minimal polynomial.

Problem 9. (1) *Show that*

$$\beta = \frac{-nb}{\gamma + (n-1)a}$$

and conclude that the minimal polynomial has degree n.

(2) *Show that the minimal polynomial of γ is*

$$(x + (n-1)a)^n - na(x + (n-1)a)^{n-1} + (-1)^n b^{n-1}.$$

(3) *Show that* $\mathrm{disc}(f) = (-1)^{\frac{n(n-1)}{2}} \left(n^n b^{n-1} + (-1)^{n-1}(n-1)^{n-1}a^n\right)$.

Some special cases: $\mathrm{disc}(x^3 + px + x) = -4p^3 - 27q^3$, and $\mathrm{disc}(x^4 + px + q) = -27p^4 + 256q^3$. Note that discriminant can be defined for any polynomial (irreducible or not). But the above calculation requires f is irreducible because Problem 8 required so. Actually, the same formula holds even it is reducible.