

HOMEWORK 14

Due date:

Exercise 7.1, ($\delta = \sqrt{-5}$ in problem 7.1), 7.3, (in exercise 7.3, R is the integer ring of $\mathbb{Q}(\sqrt{-26})$), Let F be an algebraic number field and \mathcal{O}_F be its ring of integers. Let $\mathfrak{a} \subset \mathcal{O}_F$ be a nonzero ideal. Recall that $\mathcal{O}_F/\mathfrak{a}$ is finite. We have defined

$$\text{Nm}(\mathfrak{a}) = |\mathcal{O}_F/\mathfrak{a}|,$$

which is a positive integer.

Problem 1. Suppose $[F : \mathbb{Q}] = n$. For $a \in \mathbb{Z}$, show that $\text{Nm}(a\mathcal{O}_F) = a^n = \text{Nm}_{F/\mathbb{Q}}(a)$.

This is proved in class. Repeat it here.

Problem 2. Let $\mathfrak{a}, \mathfrak{b}$ are two coprime nonzero ideals. Show that $\text{Nm}(\mathfrak{a}\mathfrak{b}) = \text{Nm}(\mathfrak{a})\text{Nm}(\mathfrak{b})$.

This is a consequence of Chinese remainder theorem.

Problem 3. Let \mathfrak{p} be a nonzero prime ideal of \mathcal{O}_F . Show that $\text{Nm}(\mathfrak{p}) = p^f$ for some prime integer $p \in \mathbb{Z}$ and some positive integer f .

Problem 4. Let F be an algebraic number field and \mathcal{O}_F be its ring of integers. From Hurwitz lemma, there exists an integer $M = M_F$ such that for any $\alpha, \beta \in \mathcal{O}_F$ with $\beta \neq 0$, there exists an integer t with $1 \leq t \leq M$ and an element $\omega \in \mathcal{O}_F$ such that

$$|\text{Nm}_{F/\mathbb{Q}}(t\alpha - \omega\beta)| < |\text{Nm}_{F/\mathbb{Q}}(\beta)|.$$

Re-examine the proof given in class and try to find an explicit form of M_F . For the field $F = \mathbb{Q}(\sqrt{-13})$, find an explicit $M = M_F$. The constant M should be as small as possible.

We know that $M_F > 1$ since \mathcal{O}_F is not a PID when $F = \mathbb{Q}(\sqrt{-13})$. Is $M = 2$ enough? If so, prove it. If not, find one counter example and try the next one.

Problem 5. Let F be an algebraic number field. Show that \mathcal{O}_F is a PID iff for every $\alpha \in F, \alpha \notin \mathcal{O}_F$, there exists $\beta, \gamma \in \mathcal{O}_F$ such that

$$0 < |\text{Nm}_{F/\mathbb{Q}}(\alpha\beta - \gamma)| < 1.$$

Problem 6. Let K be an algebraic number field. Show that there exists a finite extension L/K such that for every ideal $\mathfrak{a} \subset \mathcal{O}_K$, the ideal $\mathfrak{a}\mathcal{O}_L$ is principal in \mathcal{O}_L .

Hint: use finiteness of class numbers. See [this link](#) for a solution.

Given a matrix $A = (a_{i,j}) \in \text{Mat}_{n \times n}(\mathbb{C})$, recall that the Hilbert-Schmidt norm is defined to be

$$\|A\|_{HS} = \sqrt{\sum_{i,j} |a_{i,j}|^2}.$$

Problem 7. Show that

$$\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}$$

for all $A, B \in \text{Mat}_{n \times n}(\mathbb{C})$.