## **HOMEWORK 4**

Due date: Tuesday of Week 5

Exercises: 4.1, 4.2, 4.3, 4.5, 4.6, 4.9, 4.16, 4.17, 4.18, 5.1, 5.2, 5.3, 5.6, 5.7, 5.10, pages 380-381, Exercises: 1.1, 1.3, 1.4, 2.1, 2.2, 2.4, page 437.

Exercise 5.9, page 381 is hard. You can give it a try. For Ex.1.3, page 437, the answer should be  $m^n$ , where n is the degree of  $\alpha$ , or the degree of its minimal polynomial. For Ex. 2.1, page 437, the answer is no. Think about the reason. Ex. 2.4 page 437 is related to Exercise 2.1 page 437.

**Problem 1.** Let p be a prime integer. Show that the polynomial  $f = x^7 - p^2 \in \mathbb{Q}[x]$  is irreducible.

Hint: You can imitate the proof of Eisenstein criterion.

**Problem 2.** Let  $\mathfrak{p}$  be a nonzero prime ideal of the Gauss integer ring  $\mathbb{Z}[i]$ . Show that  $\mathbb{Z}[i]/\mathfrak{p}$  is a field and thus  $\mathfrak{p}$  is a maximal ideal. Determine the order of  $\mathbb{Z}[i]/\mathfrak{p}$ .

Hint: Show that  $\mathbb{Z}[i]/\mathfrak{p}$  is finite integral domain and thus is a field. See Ex.7.1, page 357 and Ex. 1.3, page 437.

**Problem 3.** Consider the polynomial  $f = x^4 - 10x^2 + 1 \in \mathbb{Z}[x]$ . Show that f is irreducible. Moreover, for each prime p, show that  $\psi_p(f) \in \mathbb{F}_p[x]$  is reducible,

Hint: See this link and this link.

Let R be a commutative ring and let M be an R-module. For  $m \in M$ , define  $Ann(m) = \{r \in R : rm = 0\}$ .

**Problem 4.** Show that  $\operatorname{Ann}(m)$  is an ideal of R. Moreover, the map  $\phi : R \to M$  defined by  $\phi(r) = rm$  defines an isomorphism  $R/\operatorname{Ann}(m) \cong \langle m \rangle$ , where  $\langle m \rangle$  denotes the submodule of M generated by m, namely,  $\langle m \rangle = \{rm : r \in R\}$ .

In particular, if M = R, then  $R/Ann(a) \cong (a)$ , where (a) denotes the principal ideal generated by a. So when is (a) a free module?

Let R be a ring and let  $S \subset R$  be a multiplicative set (which means  $1 \in S$ , and if  $a, b \in S$ , then  $ab \in S$ ). We have defined  $S^{-1}R$ . This construction can also be defined on modules. Let M be an R-module and we define an equivalence relation on  $M \times S$  by

 $(m,s) \sim (n,t) \iff u(tm-sn) = 0$ , for some  $u \in S$ .

**Problem 5.** Show that  $\sim$  is an equivalence relation.

Let m/s denotes the equivalence class of (m, s) for  $m \in M, s \in S$  and let  $S^{-1}M$  denotes the set of all equivalence classes. Define an abelian group structure on  $S^{-1}M$  by

$$m/s + n/t = (tm + sn)/(st).$$

Define a module structure on  $S^{-1}M$  over the ring  $S^{-1}R$  by

$$(r/s)(m/t) = (rm)/(st), r \in R, s, t \in S, m \in M.$$

Check that the above definitions are well-defined and indeed defines a  $S^{-1}R$  module structure on  $S^{-1}M$ . If  $\mathfrak{p}$  is a prime ideal and  $S = R - \mathfrak{p}$ , then we write  $S^{-1}M$  as  $M_{\mathfrak{p}}$ , which is called the localization of M at  $\mathfrak{p}$ . When is an element  $x/s \in M_{\mathfrak{p}}$  zero?

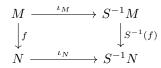
**Problem 6.** Let R be a ring  $S \subset R$  be a multiplicative subset. Let M, N be two modules.

(1) Show that the map  $\iota_M : M \to S^{-1}M$  defined by  $m \mapsto m$  is an R-module homomorphism.

(2) Suppose  $f \in \operatorname{Hom}_R(M, N)$  be a module homomorphism. Consider the map  $S^{-1}(f) : S^{-1}M \to S^{-1}N$  defined by

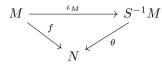
$$S^{-1}(f)(m/s) = f(m)/s.$$

Show that  $S^{-1}(f)$  is well-defined and it defines a homomorphism in  $\operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ . Moreover, the following diagram



is commutative

(3) Suppose that for any  $s \in S$ , the map  $\phi_s : N \to N$  defined by  $\phi_s(n) = sn$  is an R-module isomorphism. Given  $f \in \operatorname{Hom}_R(M, N)$ , show that there exists a unique homomorphism  $\theta : S^{-1}M \to N$  such that the following diagram



is commutative.

**Problem 7.** Let M be an R-module. Show that the following are equivalent:

- (1) M = 0;
- (2)  $M_{\mathfrak{p}} = 0$  for every prime ideal  $\mathfrak{p}$  of R;
- (3)  $M_{\mathfrak{m}} = 0$  for every maximal ideal  $\mathfrak{m}$  of R.

Hint: For (3)  $\implies$  (1), by contradiction, take  $x \in M$  and  $x \neq 0$ . Then consider  $I = \text{Ann}(x) = \{a \in R : ax = 0\}$ . The assumption  $x \neq 0$  implies that  $I \neq R$ . Thus I is contained in a maximal ideal  $\mathfrak{m}$ . By assumption, x/1 is zero in  $M_{\mathfrak{m}}$ ...

**Problem 8.** Let  $\phi : M \to N$  be a homomorphism of *R*-modules. Let  $S \subset R$  be a multiplicative subset. Define  $S^{-1}(\phi) : S^{-1}M \to S^{-1}N$  by  $S^{-1}(\phi)(m/s) = \phi(m)/s$ . See Problem 6. Show that  $\ker(S^{-1}\phi) = S^{-1} \ker(\phi)$ .

 $S = A - \mathfrak{p}$  for a prime ideal  $\mathfrak{p}$ , we write the above  $S^{-1}(\phi)$  as  $\phi_{\mathfrak{p}}$ .

**Problem 9.** Let  $\phi \in \operatorname{Hom}_R(M, N)$ . Show the following are equivalent.

- (1)  $\phi$  is injective;
- (2)  $\phi_{\mathfrak{p}}$  is injective for every prime ideal  $\mathfrak{p}$  of R;
- (3)  $\phi_{\mathfrak{m}}$  is injective for every maximal ideal  $\mathfrak{m}$  or R.

A property on modules and/or its homomorphisms is called a local property if it only depends on localizations. The above problem shows that being injective is a local property. Localization are very important tools in studying modules.

**Problem 10.** Let R be a ring and  $I \subset R$  be an ideal. Let  $f \in R$ . Suppose that  $f \in IR_{\mathfrak{m}}$  for every maximal ideal  $\mathfrak{m}$ , show that  $f \in I$ . Here  $IR_{\mathfrak{m}} = \{ax : a \in I, x \in R_{\mathfrak{m}}\}$ .

The condition  $f \in IR_{\mathfrak{p}}$  says that f is locally in I at the prime  $\mathfrak{p}$ . Hint: There are two ways to prove the above. (1) Consider the quotient ring R/I and use an argument similar to Problem 7. (2) The assumption  $f \in IR_{\mathfrak{m}}$  says that there exists  $a_{\mathfrak{m}} \in I$  and  $s_{\mathfrak{m}} \notin \mathfrak{m}$  such that  $f = a_{\mathfrak{m}}/s_{\mathfrak{m}}$  for each  $\mathfrak{m}$ . Consider the ideal J generated by  $\{s_{\mathfrak{m}}\}$  as  $\mathfrak{m}$  runs through all maximal ideals. Show that J = R. The result should follow easily.

The following is an analogue/generalization. Let R be a ring and  $\mathfrak{a} \subset R$  be an ideal. Let M be an R-module and let  $f \in M$ . Suppose that  $f \in \mathfrak{a}M_p$  for every prime ideal  $\mathfrak{p}$ , is it true that  $f \in \mathfrak{a}M$ ? Here  $\mathfrak{a}M = \left\{ \sum_{finite} a_i m_i : a_i \in \mathfrak{a}, m_i \in M \right\}$ , which is a submodule of M. Actually, one can also pass to the quotient  $M/\mathfrak{a}M$ . Now a natural question is: is it true that  $(M/\mathfrak{a}M)_p \cong M_p/\mathfrak{a}M_p$ ? The answer should be yes and the proof is not hard.