HOMEWORK 5

Due date: Tuesday of Week 6,

Exercises: 2.3, 6.1, 6.2, page 437-439

Here is a translation of Ex 6.1. Let $f_1, f_2, f_3, \ldots, \in \mathbb{C}[x_1, \ldots, x_n]$ be an infinite sequence of polynomials. Let $V = \{a = (a_1, \ldots, a_n) \in \mathbb{C}^n : f_i(a) = 0, \forall i \geq 1\}$. Show that there exists a finite number of polynomials $g_1, \ldots, g_k \in \mathbb{C}[x_1, \ldots, x_n]$ such that $V = \{a = (a_1, \ldots, a_n) \in \mathbb{C}^n : g_i(a) = 0, 1 \leq i \leq k\}$.

Problem 1. *Suppose that the following diagram of R-modules is commutative and the rows are exact sequences*

$$\begin{tikzcd} & M \arrow[r, "\phi"] \arrow[d, "f"] & N \arrow[r, "\psi"] \arrow[d, "g"] & P \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\phi'"] & N' \arrow[r, "\psi'"] & P' & \end{tikzcd}$$

Show that there is an exact sequence

$$0 \to \operatorname{Ker}(\phi) \to \operatorname{Ker}(f) \to \operatorname{Ker}(g) \to \operatorname{Ker}(h) \to \operatorname{Coker}(f) \to \operatorname{Coker}(g) \to \operatorname{Coker}(h) \to \operatorname{Coker}(\psi') \to 0.$$

The homomorphism $\partial : \operatorname{Ker}(h) \to \operatorname{Coker}(f)$ is defined as

$$\partial(x) = (\phi')^{-1} g \psi^{-1}(x), \forall x \in \operatorname{Ker}(h).$$

You are required to check that ∂ is well-defined and check the sequence is exact at every place.

The above assertion is called the extended snake lemma.

Problem 2. *Use the above extended snake lemma to give a new proof of the following 3-lemma. Given the following commutative diagram of R-modules with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, "f"] & N \arrow[r, "\psi"] \arrow[d, "g"] & P \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\phi'"] & N' \arrow[r, "\psi'"] & P' \arrow[r] & 0 \end{tikzcd}$$

If two of f, g, h are isomorphisms, then the third one must be an isomorphism.

Think about if it is possible to use the above 3 lemma to give a new proof of the 5-lemma.

Problem 3. *Let*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0 \tag{0.1}$$

be a sequence of R-modules. Let N be another R-module. We can form the following sequence

$$0 \longrightarrow \operatorname{Hom}(M'', N) \xrightarrow{v^*} \operatorname{Hom}(M, N) \xrightarrow{u^*} \operatorname{Hom}(M', N) \tag{0.2}$$

where $u^*(f) = f \circ u$ for $f \in \operatorname{Hom}(M, N)$ and v^* is defined similarly. Show that the sequence (0.1) is exact iff the sequence (0.2) is exact for any R-module N.

Problem 4. *Let*

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N'' \tag{0.3}$$

be a sequence of R-modules. Let M be another R-module. We can form the following sequence

$$0 \longrightarrow \operatorname{Hom}(M, N') \xrightarrow{u_*} \operatorname{Hom}(M, N) \xrightarrow{v_*} \operatorname{Hom}(M, N'') \tag{0.4}$$

where $u_*(f) = u \circ f$ for $f \in \operatorname{Hom}(M, N')$ and v_* is defined similarly. Show that the sequence (0.3) is exact iff the sequence (0.4) is exact for any R-module M.

An R-module M is called Noetherian if its submodules satisfies acc conditions. Recall that this is equivalent to that every submodule of M is finitely generated. We learned in class that a finitely generated module over a Noetherian ring is a Noetherian module.

Problem 5. *Given a short exact sequence of R-modules*

$$0 \to N \to M \to P \to 0.$$

Show that M is Noetherian iff both N and P are Noetherian.

Caution: A subring of a Noetherian ring is not necessarily a Noetherian ring. Example, let $R = F[x_1, x_2, \dots]$ be a polynomial ring with infinite number of variables over a field F. You can check that R is an integral domain but not Noetherian. Let K be the fractional field of R. Then K is a Noetherian ring but R is not.

Problem 6. *Let R be a ring and M be a Noetherian R-module. Let $f \in \mathrm{Hom}_R(M, M)$. Show that if f is surjective then it is also injective and thus an isomorphism. Give an example of $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ (here $\mathbb{Z}$ is viewed as an $\mathbb{Z}$-module and thus f is just a homomorphism between abelian groups, which is not required to be a ring homomorphism) such that f is injective but it is not surjective.*

Hint: consider the chain submodules $\mathrm{Ker}(f) \subset \mathrm{Ker}(f^2) \subset \cdots \subset \mathrm{Ker}(f^n) \subset \dots$. Since M is Noetherian, it is stationary. Let $N = \cup_{i=1}^{\infty} \mathrm{Ker}(f^i) = \mathrm{Ker}(f^n)$ for a big n. Consider the map $g = f|_N : N \to N$. Show that g is well-defined (namely $g(N)$ is indeed in N) and surjective. Then consider g^n.

If M is Noetherian (which means any submodule of M is finitely generated), then it is finitely generated. The converse is false. Namely, if M is finitely generated, it is not necessarily Noetherian, unless the ring R is also Noetherian. For example, if $R = F[x_1, x_2, \dots,]$, the polynomial ring over a field F with infinitely many variables, and $M = R$. Then M is finitely generated (actually it is generated by $1 \in R$), but M is not Noetherian, because its submodule (in this case, it is just an ideal) $I = \langle x_1, x_2, \dots, \rangle$ is not finitely generated.

The above problem is also true if M is finitely generated (without assuming it is Noetherian), which you can prove in the next HW. But it is false if M is not finitely generated, namely, if R is not finitely generated R-module, then $f \in \mathrm{Hom}_R(M, M)$ surjective does not imply it is injective. Even when R is a field so every module is a vector space, surjectivity does not imply injectivity if the dimension is not finite. Here is one example. Consider $M = \mathbb{Z} \times \mathbb{Z} \times \cdots = \prod_{i \geq 0} \mathbb{Z}$ (infinite copies of $\mathbb{Z}$, see also the next part of this HW for infinite product of modules) and consider the map

$$f : M \to M$$

$$(\alpha_0, \alpha_1, \alpha_2, \dots) \mapsto (\alpha_1, \alpha_2, \alpha_3, \dots).$$

Then f is surjective but $\mathrm{Ker}(f) \cong \mathbb{Z}$. On the other hand, the isomorphism $M/\mathrm{Ker}(f) \cong \mathrm{Im}(f) = M$ does not directly imply $\mathrm{Ker}(f) = 0$. See the above example when $M = \prod_{i \geq 0} \mathbb{Z}$. Think about why (the reason is: equality and isomorphism are different. This is subtle but important).

1. Product and direct sum of modules

We have defined direct sum and direct product of vector spaces. The definition can be extended to modules over rings. We fix a ring R (which is assumed to be commutative with 1 as usual).

Definition 1. Let I be an index set. Suppose that we are given a module M_i over R for each $i \in I$. The **direct sum** of M_i is a pair $(\oplus_{i \in I} M_i, (\iota_i)_{i \in I})$, where

(1) $\oplus_{i \in I} M_i$ is a module over R; and
(2) $(\iota_i)_{i \in I}$ is a family of maps $\iota_i \in \mathrm{Hom}_R(M_i, \oplus_{i \in I} M_i)$,

such that for any other R-module X, and for any other family of linear maps $f_i \in \mathrm{Hom}_R(M_i, X)$ for each $i \in I$, there is a unique homomorphism $f : \oplus_{i \in I} M_i \to X$ such that $f_i = f \circ \iota_i$ for each $i \in I$.

In other words, we have the following diagram

$$\begin{tikzcd} M_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & \oplus_{i\in I} M_i \arrow[dl, dashed, "\exists! f"] \\ & X & \end{tikzcd}$$

Dually, we can define direct products.

Definition 2. Let I be an index set. Suppose that we are given a module M_i over R for each $i \in I$. The **direct product** of M_i is a pair $(\prod_{i\in I} W_i, (p_i)_i)$, where

(1) $\prod_{i\in I} M_i$ is an R-module; and
(2) $p_i : \prod_{i\in I} M_i \to M_i$ is an R-module homomorphism,

satisfying the following universal property. Given any pair $(X, (f_i)_{i\in I})$, where X is an R-module and $f_i \in \mathrm{Hom}_R(X, M_i)$, then there is a unique linear map $f : X \to \prod_{i\in I} M_i$ such that $f_i = p \circ f$.

$$\begin{tikzcd} M_i & & \prod_{i\in I} M_i \arrow[ll, "p_i"'] \\ & X \arrow[ul, "f_i"] \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}$$

Theorem 1.1. *Direct product and direct sum exists.*

Proof. Define $\prod_{i\in I} M_i$ as follows. As a set, it is just the product whose elements are $(\alpha_i)_{i\in I}$, with each $\alpha_i \in M_i$; the addition and scaler product are defined component wise:

$$(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} := (\alpha_i + \beta_i)_{i\in I}, \alpha_i, \beta_i \in M_i;$$
$$r(\alpha_i)_{i\in I} := (r\alpha_i)_{i\in I}, r \in R.$$

Define $p_i : \prod_{i\in I} M_i \to M_i$ be the projection: $p_i((\alpha_i)_{i\in I}) = \alpha_i$.

Define $\oplus_{i\in I} M_i = \left\{(\alpha_i)_{i\in I} \in \prod_{i\in I} M_i | \ \alpha_i = 0 \text{ except for a finite number of } i \in I\right\}$ and define $\iota_i : M_i \to \oplus_{i\in I} M_i$ by $\iota_i(\alpha) = (\alpha_i)_{i\in I}$, where $\alpha_i = \alpha$ and $\alpha_j = 0$ if $j \neq i$. $\square$

Problem 7. *Show that $(\prod_{i\in I} M_i, (p_i)_{i\in I})$ is indeed the direct product (namely, it satisfies the universal property in Definition 2) and $(\oplus_{i\in I} M_i, (\iota_i)_{i\in I})$ satisfies the universal property in Definition 1. Thus the direct sum is the same with direct product only when the index set I is finite.*

Problem 8. *Let I be an index set. Suppose that we are given a module M_i over R for each $i \in I$. Let X be any R-module. Show that there are isomorphisms*

$$\mathrm{Hom}_R(X, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Hom}_R(X, M_i);$$
$$\mathrm{Hom}_R(\oplus_{i\in I} M_i, X) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, X).$$

Keep in mind that $\mathrm{Hom}_R(X, M_i)$ is also an R-module and $\prod_{i\in I} \mathrm{Hom}_R(X, M_i)$ denotes product of R-modules.

Let M be an R-module. Recall that a subset $S \subset M$ is called a basis of S if (1) S is linearly independent and (2) S generates M. An R-module M is called free if it has a basis. If $|S| = n$, we get an isomorphism $R^n \cong M$.

Problem 9. *Let I be an index set and M_i is a free R-module for each i. Show that $\oplus_{i\in I} M_i$ is also a free R-module.*

In general, an infinite direct product of free modules is not free. For example, $\prod_{i\in I} \mathbb{Z}$ is not a free abelian group if I is infinite, see this link.

Problem 10. *Let R be a ring and N be any R-module. Show that there exists an isomorphism*

$$\mathrm{Hom}_R(R^m, N) \cong N^m,$$

for a positive integer m.

If R is a field. This is Theorem 3.1 of Hoffman-Kunze. For general R the proof is similar. It can also be deduced from Problem 8.