HOMEWORK 6

Due date: Tuesday of Week 7
Exercises: 4.1, 4.3, 4.4, 4.6, 4.7, 4.8, page 438 of Artin's book.

Let R be a PID, let $A = (a_{ij}) \in \mathrm{Mat}_{m\times n}(R)$ be a matrix. Given subsets $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$, such that $|I| = |J| = k$. We assume that

$$I = \{i_1, \ldots, i_k\}, 1 \le i_1 < \cdots < i_k \le m,$$
$$J = \{j_1, \ldots, j_k\}, 1 \le j_1 < \cdots < j_k \le n.$$

We consider the submatrix $A_{I,J}$ of A defined by

$$A = \begin{bmatrix} a_{i_1 j_1} & \ldots & a_{i_1 j_k} \\ \vdots & & \vdots \\ a_{i_k j_1} & \ldots & a_{i_k j_k} \end{bmatrix},$$

and $D_{I,J}(A) = \det(A_{I,J})$. Recall that we have defined 4 elementary row (and column) operations by multiplying elementary matrix. A type I elementary matrix is obtained by multiplying an element $c \in K^\times$ to a row of I_n, which is denoted by $E_n(R_i \leftarrow cR_i)$. Here as usual, I_n is the identity matrix. A type II elementary matrix is obtained by adding cR_j to R_i of the identity matrix I_n for some $c \in K$, which is denoted by $E_n(R_i \leftarrow R_i + cR_j)$. A type III elementary matrix is obtained by switching two rows of I_n, which is denoted by $E_n(R_i \leftrightarrow R_j)$. Type 4 elementary matrix is of the form

$$\begin{bmatrix} a & b & & & \\ c & d & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, ad - bc = 1, a, b, c, d \in R.$$

Problem 1. *Let e be an elementary operation of one type defined above. For $A \in \mathrm{Mat}_{m\times n}(R)$. Denote*

$$\Delta_k(A) = gcd\left\{D_{I,J}(A) : |I| = |J| = k\right\}.$$

Show that $(\Delta_k(A)) = (\Delta_k(e(A)))$ for any k. Here the equality $(\Delta_k(A)) = (\Delta_k(e(A)))$ is an equation of two principal ideals.

This is Theorem 10 page 259 of Hoffman-Kunze if e is of the first 3 types. You only need to check the 4th type elementary operation.

Let R be a ring and let M be an R-module. Recall that M is called finitely presented (or it has a finite presentation), if there exists an exact sequence

$$R^n \to R^m \to M \to 0,$$

for some non-negative integers m and n. Equivalently, M is finitely presented if there exists a surjection $\varphi : F^m \to M$ such that $\ker(\varphi)$ is finitely generated.

Problem 2. *Let R be a ring and M be a finitely presented module. Let $f : R^k \to M$ be any surjective map. Show that* $\mathrm{Ker}(f)$ *is finitely generated.*

Note that the assumption says that there exists a surjection $\varphi : F^m \to M$ such that $\mathrm{Ker}(\varphi)$ is finitely generated. The assertion says that for any surjection of the form $f : R^k \to M$, its kernel is always finitely generated. Hint: See this link for a proof.

The following is a very typical example on how to use finite presentation. Let R be a ring and M, N be two R-modules. Recall that $\mathrm{Hom}_R(M,N)$ also has an R-module structure. Let $\mathfrak{p}$ be a prime ideal of R. We define a map

$$\theta_{M,N} : (\mathrm{Hom}_R(M,N))_{\mathfrak{p}} \to \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

as follows. First for $f \in \mathrm{Hom}_R(M,N)$, we have a homomorphism $S^{-1}(f) \in \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ as in HW4, problem 5. Here $S = A - \mathfrak{p}$.

Problem 3. *Let the notations be as above.*

(1) *Show that the map*

$$\mathrm{Hom}_R(M,N) \ni f \mapsto S^{-1}(f) \in \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

can be uniquely extended to $(\mathrm{Hom}_R(M,N))_{\mathfrak{p}}$*, namely, there is a unique homomorphism* $\theta_{M,N} : (\mathrm{Hom}_R(M,N))_{\mathfrak{p}} \to \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Hom}_R(M,N) \arrow[rr] \arrow[dr, "S^{-1}(\cdot)"] & & (\mathrm{Hom}_R(M,N))_{\mathfrak{p}} \arrow[dl, "\theta_{M,N}"] \\ & \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) & \end{tikzcd}$$

is commutative.

(2) *Suppose that* $M = R^m$ *for a positive integer* m*, show that* $\theta_{F^m,N}$ *is an isomorphism.*

(3) *Suppose that* M *is finitely presented, show that* $\theta_{M,N}$ *is an isomorphism.*

Hint: Part (1) follows from HW4, problem 5. Part (2) follows from Problem 10, HW5. For (3), use a commutative diagram.

Problem 4. *Let* R *be a ring and* $I \subset R$ *be an ideal. Let* M *be a finitely generated* R*-module such that* $IM = M$ *(where* $IM = \{\sum a_i m_i : a_i \in I, m_i \in M\}$*.) Show that there exists an element* $a \in I$ *such that* $m = am$ *for any* $m \in M$*.*

The assertion of Problem 4 is called Nakayama's lemma. Hint: Let $\{m_1, \ldots, m_n\} \subset M$ be a set of generators. The assumption $M = IM$ says that $m_i = \sum a_{ij} m_j$ with $a_{ij} \in I$. In other words, there is a matrix $A \in \mathrm{Mat}_{n\times n}(I)$ such that $X = AX$, or $(I_n - A)X = 0$. where $X = [m_1, \ldots, m_n]^t$ and I_n is the identity matrix. Now multiply both sides by the classical adjoint of $I_n - A$.

Problem 5. *Let* R *be a local ring with unique maximal ideal* $\mathfrak{m}$*. Let* M *be a finitely generated* R*-module such that* $\mathfrak{m}M = M$*. Show that* $M = 0$*.*

Hint: This is a Corollary of Problem 4.

Problem 6. *Let* R *be a ring and* M *be a finitely generated* R*-module. Let* $T \in \mathrm{Hom}_R(M,M)$ *be a surjective homomorphism. Show that* T *is injective.*

Hint: View M as an $R[x]$ module via $f(x).m := f(T)m$. We did this many times in Linear algebra. Clearly, M is also a finitely generated $R[x]$-module. Consider the ideal $I = xR[x] \subset R[x]$ of $R[x]$. Since T is surjective, $M = IM$. Now apply Nakayama's lemma.

Problem 7. *Let* R *be a ring and let* M, N *be two* R*-modules. Given two surjective* $T_1, T_2 \in \mathrm{Hom}_R(M,N)$*.*

(1) *Show that* $\ker(T_1) \subset \ker(T_2)$ *if and only if there exists a homomorphism* $\phi : N \to N$ *such that* $\phi \circ T_1 = T_2$*.*

(2) *Show that* $\ker(T_1) = \ker(T_2)$ *if and only if there exists a isomorphism* $\phi : N \to N$ *such that* $\phi \circ T_1 = T_2$*.*

Hint: Draw two exact sequences.

If R in the above problem is a field, we can drop the condition that T_1, T_2 are surjective.

Problem 8. *Let* F *be a field and let* V, W *be two finite dimensional* F*-vector spaces. Given two* $T_1, T_2 \in \mathrm{Hom}_F(V,W)$*.*

(1) *Show that* $\ker(T_1) \subset \ker(T_2)$ *if and only if there exists a homomorphism* $\phi : W \to W$ *such that* $\phi \circ T_1 = T_2$.
(2) *Show that* $\ker(T_1) = \ker(T_2)$ *if and only if there exists a isomorphism* $\phi : W \to W$ *such that* $\phi \circ T_1 = T_2$.

Here is a dual version of the above problem.

Problem 9. *Let R be a module and let V, W be two R modules. Suppose that V is a free R-module. Given two $T_1, T_2 \in \mathrm{Hom}_F(V, W)$. Show that* $\mathrm{Im}(T_1) \subset \mathrm{Im}(T_2)$ *if and only if there exists a homomorphism $\phi : V \to V$ such that $T_1 = T_2 \circ \phi$.*

The last two problems were final exam problems of 2023.