HOMEWORK 7

Due date: Tuesday of Week 8

Exercises: 7.1, 7.2, 7.5, 7.8, 7.9, 8.6, page 439-440 of Artin's book; Exercises: 1.1, 1.2, 2.1, 2.3, page 472.

Problem 1. *Consider the abelian group*

$$A = C_{30} \oplus C_{49} \oplus C_{12} \oplus C_{25} \oplus C_{40}.$$

Find the invariant divisors of A. Here C_n denotes the cyclic groups of order n.

Problem 2. *Let R be a PID and M be a free R-module of rank m. Let N be a submodule of M. We know that N is a free module of rank n with $n \le m$. Show that there exists a basis $\mathcal{B} = \{e_1, \ldots, e_m\}$ of M and non-zero elements $a_1, \ldots, a_n \in R$ such that:*

1. *the elements $a_1e_1, a_2e_2, \ldots, a_ne_n$ form a basis of N over R;*
2. *we have $a_i | a_{i+1}$ for $i = 1, \ldots, n-1$.*

The sequence of ideas $(a_1), \ldots, (a_n)$ is uniquely determined by the above conditions.

Problem 3. *Let $G = \mathrm{GL}_2(\mathbb{Q})$ and $H = \mathrm{GL}_2(\mathbb{Z})$. Determine the double coset*

$$H \backslash G / H.$$

Problem 4. *Let R be a commutative nonzero ring with 1. Let m, n be two positive integers. If there is an injective R-module homomorphism $f : R^m \to R^n$, show that $m \le n$.*

Hint: By contradiction, suppose that $m > n$. Let $\iota : R^n \to R^m$ be the natural inclusion. Namely, $\iota(r_1, \ldots, r_n) = (r_1, \ldots, r_n, 0, \ldots, 0)$. Let $\phi = \iota \circ f : R^m \to R^m$, which is also injective. Let $\pi : R^m \to R$ be the projection to the last coordinate. Then $\pi \circ \phi = 0$. Let $\chi \in R[x]$ be the characteristic polynomial of ϕ. Then Cayley-Hamilton says that $\chi(\phi) = 0$. Think about why Cayley-Hamilton is still true (try to repeat the proof!) Assume that $\chi = X^k P$ for a polynomial $P = x^{m-k} + \cdots + c_0$ with $c_0 \ne 0$. Then $\chi(\phi) = 0$ implies that $\phi^k P(\phi) = 0$. Show that $P(\phi) = 0$ and thus $\pi \circ P(\phi) = 0$. Then get a contradiction.

Problem 5. *Let K/F be a field extension and $\alpha \in K$ is algebraic over F with $\deg(\alpha) = d$. Show that $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is a basis of $F[\alpha]/F$.*

1. Linear operators and f.g. modules over PID

In this section, let F be a field and V be a finite dimensional vector space over F. Let $T : V \to V$ be a linear operator. We can view V as an $F[x]$-module by $f(x).v := f(T)v$ for any $f \in F[x]$.

Problem 6.
1. *Show that a subspace $W \subset V$ is T-invariant iff W is a submodule of V;*
2. *Show that V has a cyclic vector iff V can be generated by a single element as an $F[x]$-module.*

Recall that a subspace $W \subset V$ is called T-admissible if (1) W is T-invariant; and (2) if $f(T)\beta \in W$ for $\beta \in V, f \in F[x]$, then there exists a vector $\gamma \in W$ such that $f(T)\beta = f(T)\gamma$. See Section 7.2 of Hoffman-Kunze. The cyclic decomposition theorem (Theorem 7.3 and its corollary of Hoffman-Kunze) said that W is T-admissible iff there exists another T-invariant subspace W' such that $V = W \oplus W'$.

The following are some generalizations of the above terminology into more general modules. Let R be a general ring and let M be an R-module. A submodule N of M is called a **direct summand** of M if there exists another submodule N' of M such that $M = N \oplus N'$. This is a generalization that *there exists another T-invariant subspace W' such that $V = W \oplus W'$.*

A submodule N of M is called **pure** if for any $m \times n$ matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n} \in \mathrm{Mat}_{m \times n}(R)$, and any element $Y = (y_1, \ldots, y_m)^t$ with $y_i \in N$, if there exist $X = (x_1, \ldots, x_n)^t$ with $x_i \in M$ such that

$$AX = Y,$$

then there exists $X' = (x'_1, \ldots, x'_n)^t$ with $x'_i \in N$ such that

$$AX' = Y.$$

The definition of pure submodule looks complicate. Here is a digression.

Problem 7. *Suppose that N is a pure submodule of M and there is a commutative diagram of R-modules*

$$\begin{array}{ccccc} & & R^n & \xrightarrow{f} & R^m \\ & & \downarrow u & & \downarrow v \\ 0 & \longrightarrow & N & \xrightarrow{i} & M \end{array}$$

Here m, n are positive integers and $i : N \to M$ denotes the inclusion. Show that there is homomorphism $\phi : R^m \to N$ such that $u = \phi \circ f$. (We don't require $v = i \circ \phi$.)

Let ϵ_i be the standard basis of R^n and e_j be the standard basis of R^m. Then $vf(\epsilon_i) = iu(\epsilon_i) \in N$. Or $v(\sum a_{ij} e_j) = \sum_j a_{ij} v(e_j) \in N$. By definition, there exists $x'_j \in N$ such that $\sum a_{ij} x'_j = \sum a_{ij} v(e_j)$. Define $\phi(e_j) = x'_j$. Then $\phi \circ f(\epsilon_i) = \sum a_{ij} x'_j = vf(\epsilon_i) = u(\epsilon_i)$. Thus $u = \phi \circ f$.

Let M be an R-module, a submodule $N < M$ is called **admissible** if for any $r \in R$ and $x \in M$ if $rx \in N$, then there exists an $n \in N$ such that $rx = rn$. This agrees with the notation defined in Hoffman-Kunze when $R = F[x]$ and $M = V$. Note that, a pure submodule is admissible (since pure requires a condition for any $m \times n$).

Problem 8. *Let R be a PID. Let M be an R-module and $N < M$ be a submodule. Show that N is a pure submodule iff it is an admissible submodule.*

Hint: You need to show any admissible submodule is pure. Use diagonalization. This is not hard.

Problem 9. *Let R be a ring and M be an R-module. Let N be a submodule of M. Consider the short exact sequence*

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow 0\ .$$

Show that the following are equivalent

1. *N is a direct summand of M;*
2. *there exists a homomorphism $s \in \mathrm{Hom}_R(M, N)$ such that $s(x) = x$ for all $x \in N$ (namely, $s \circ i = \mathrm{id}_N$);*
3. *for each R-module P, the sequence*

$$0 \to \mathrm{Hom}_R(M/N, P) \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(N, P) \to 0$$

   is exact;
4. *there exists a homomorphism $u \in \mathrm{Hom}_R(M/N, M)$ such that $\pi \circ u = \mathrm{id}_{M/N}$;*
5. *for each R-module P, the sequence*

$$0 \to \mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M/N) \to 0$$

   is exact.

This is called the splitting lemma. See Problems 3 and 4 of HW 5. Hint: Show $(1) \Longrightarrow [(2) \Longleftrightarrow (3)] \Longrightarrow [(4) \Longleftrightarrow (5)] \Longrightarrow (1)$.

Problem 10. *Let M be an R-module and let $N \subset M$ be a submodule. If N is a direct summand of M, show that N is a pure submodule.*

Problem 11. *Let R be a general ring and M be an R-module. Let $N < M$ be a pure submodule and X is a finitely presented R-modules. Show that the sequence the sequence*

$$0 \to \mathrm{Hom}_R(X, N) \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, M/N) \to 0$$

is exact. As a consequence, show that if M/N is finitely presented, then N is a pure submodule of M iff it is a direct summand.

Hint: This one might be hard. One only needs to show that $\mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, M/N)$ is surjective. Given $w \in \mathrm{Hom}_R(X, M/N)$, try to produce a commutative diagram

$$\begin{tikzcd} & R^n \arrow[r, "f"] \arrow[d, "u"] & R^m \arrow[r] \arrow[d, "v"] & X \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "\pi"] & M/N \arrow[r] & 0 \end{tikzcd}$$

and use Problem 7 to get a hom $\phi : R^m \to N$. Then consider $\widetilde{w} \in \mathrm{Hom}_R(R^m, M)$ defined by $\widetilde{w} = v - \phi$.

Problem 12. (1) *Let R be a Noetherian ring and M be a finitely generated R-module. Show that a submodule $N < M$ is pure iff it is a direct summand.*

(2) *Let R be a PID and M be a finitely generated R-module. Show that a submodule $N < M$ is admissible iff it is a direct summand.*

This problem together with the structure theorem of finite generated modules over PID fully covers Theorem 3, page 233 of Hoffman-Kunze. In the general case, we have

$$\text{(direct summand submodules)} \subset \text{(pure submodules)} \subset \text{(admissible submodules)}.$$

See this link for an example of pure submodule which is not a direct summand.

2. Presentation of linear operator as $F[x]$-modules

This problem is from HW11, 2023. It is also Exercise 8.4, page 440 of Artin's book. Try it again. It is better to copy your solution from last year here.

Let F be a field. We consider $K = F[x]$ and K^n. An element $u \in K^n$ will be considered as a column vector and thus it has the form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

and each $u_i \in F[x]$ can be written as $u_i = u_{i0} + u_{i1}x + u_{i2}x^2 + \cdots + u_{ik}x^k$ with $u_{ij} \in F$. Since u_{ik} can be zero, we can take a k such that it works for all i, namely each u_i has its last term of the form $u_{ik}x^k$. Thus we can write u as

$$u = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} x + \cdots + \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} x^k.$$

Write

$$\mathbf{u}_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix} \in F^n,$$

then we can write $u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k$. Here we write x^j in front of $\mathbf{u}_j$ (so that it looks like a scaler times a column vector). Thus an element in $K^n = F[x]^n$ can be viewed as a polynomial with coefficients in F^n.

Fix a matrix $A \in \mathrm{Mat}_{n\times n}(F)$. Note that as an element in $\mathrm{Mat}_{n\times n}(K)$, the matrix $xI_n - A$ defines a linear map $T_{(xI_n - A)} : K^n \to K^n$ defined by

$$T_{(xI_n-A)}u = (xI_n - A)u,$$

as usual. We now consider the map $\phi : K^n \to F^n$ defines as follows. Given an element

$$u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k \in K^n,$$

we define

$$\phi(u) = \mathbf{u}_0 + A\mathbf{u}_1 + \cdots + A^k\mathbf{u}_k \in F^n.$$

Namely, we just replace the symbol x by the matrix A. The notation should be clear.

Problem 13. (1) *Show that ϕ is surjective. (This should be trivial).*

(2) *Show that* $\mathrm{Im}(T_{(xI_n-A)}) \subset \ker(\phi)$. *(This is also trivial).*

(3) *Show that* $\ker(\phi) \subset \mathrm{Im}(T_{(xI_n-A)})$. *(It needs some work, but not very hard).*

The assertions of this problem say that the sequence

$$K^n \xrightarrow{T_{(xI_n-A)}} K^n \xrightarrow{\phi} F^n \longrightarrow 0$$

is exact (as K-modules), which gives a presentation of F^n as an $F[x]$-module.