HOMEWORK 12

Due date: Next Monday

Exercises 9, 11, 12, 14, 15, page 190;
Exercises 1, 2, 5, 10, 11, pages 197-198;
Exercises 4, 5, 10, 12, pages 205-206,
Exercises 1, 2, 4, 5, page 208.

You can use the result of Ex 8 of page 190 to solve Ex 9.

Problem 1. *Let F be a field and $A, B \in \mathrm{Mat}_{n\times n}(F)$ be two arbitrary square matrices. For $\lambda \in F$, consider the eigenspaces*

$$E_\lambda(AB) = \{\alpha \in F^n : AB\alpha = \lambda\alpha\} = \ker(\lambda I - AB),$$

$$E_\lambda(BA) = \{\alpha \in F^n : BA\alpha = \lambda\alpha\} = \ker(\lambda I - BA).$$

Here $I \in \mathrm{Mat}_{n\times n}(F)$ is the identity matrix.

1. *Suppose that $\lambda \neq 0$. Construct an isomorphism $E_\lambda(AB) \to E_\lambda(BA)$. (You need to show that the map you constructed is an isomorphism).*
2. *Conclude that $\mathrm{rank}(\lambda I - AB) = \mathrm{rank}(\lambda I - BA)$ if $\lambda \neq 0$ using part (1).*
3. *If $\lambda = 0$, give examples of A, B such that $\dim\ker(AB) \neq \dim\ker(BA)$ and $\mathrm{rank}(AB) \neq \mathrm{rank}(BA)$.*

(Hint for (1): For $\alpha \in E_\lambda(AB)$, what can you say about the vector $B\alpha$?)

Remark 0.1. For (1), λ is not necessarily an eigenvalue. So a special case of part (1) says that $\lambda \neq 0$ is an eigenvalue of AB if and only if λ is an eigenvalue of BA. This conclusion is also true for $\lambda = 0$. See Ex 9, page 190 and Ex 11, page 198. (Could you give a direct proof of this fact: 0 is an eigenvalue of AB if and only 0 is an eigenvalue of BA? It is not hard. So Ex 9, page 190 has a very direct proof without using Ex 8 there.) Part (2) of the above Problem gives a new proof of this fact

$$\mathrm{rank}(I - AB) = \mathrm{rank}(I - BA),$$

which we proved in the midterm exam using a different way. You might find that this proof is easier. By the way, this result is much stronger than Ex 8, page 190. Note that part (3) says that the function

$$f(\lambda) = \mathrm{rank}(\lambda I - AB) - \mathrm{rank}(\lambda I - BA)$$

might have a "jump" at the point 0, and thus is discontinuous if the field F is $\mathbb{R}$ or $\mathbb{C}$.

Remark 0.2. Given $A, B \in \mathrm{Mat}_{n\times n}(F)$. Exercise 9 page 190 shows that AB and BA have the same eigenvalues. Given an eigenvalue λ of AB (and hence of BA). The above Problem shows that the geometric multiplicity of λ for AB and BA are the same if $\lambda \neq 0$. You might be wondering if AB and BA have the same algebraic multiplicity at λ. Actually this is true for all λ because it is true that

$$\chi_{AB} = \chi_{BA}.$$

This is Exercise 11, page 198. You could find many different solutions of Exercise 11, page 198 here. A particular interesting one is given here. We will learn the formula which is cited there, namely,

$$\chi_A(x) = \det(xI_m - A) = \sum_{k=0}^{m} \mathrm{Tr}(\wedge^k(A))(-1)^k x^{m-k}$$

for $A \in \mathrm{Mat}_{m\times m}(F)$, in next semester.

Problem 2. *View $\mathbb{C}^n$ as a vector space over $\mathbb{R}$, which has dimension $2n$. Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and we consider the $\mathbb{R}$-linear map $T_A : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T_A(\alpha) = A(\alpha)$. After fixing a basis of $\mathbb{C}^n$ (viewed as an $\mathbb{R}$-vector space), we get a matrix $[T_A] \in \mathrm{Mat}_{(2n)\times(2n)}(\mathbb{R})$. Show that $\det([T_A]) = \mathrm{Nm}_{\mathbb{C}/\mathbb{R}}(\det(A))$. In particular, $\det([T_A]) \geq 0$.*

Recall that for $z \in \mathbb{C}$, $\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}(z)$ is just $z\overline{z}$. Note that the matrix $[T_A]$ depends on the choice of basis, but its determinant does not.
Hint: We can assume A is upper triangular. Why?

Problem 3. *Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Suppose that $A^2 + aA + bI_n = 0$ for some $a, b \in \mathbb{C}$ with $a^2 - 4b \neq 0$. Show that A is diagonalizable.*

Problem 4. *Consider the Fibonacci sequence $\{F_n\}_{n\geq 0}$ given by $F_0 = F_1 = 0$ and*

$$F_{n+1} = F_{n+1} + F_n, \forall n \geq 0.$$

For each $n \geq 0$, consider the vector

$$\alpha_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \in \mathbb{C}^2.$$

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

(1) *Show that $\alpha_{n+1} = A\alpha_n$ and deduce that $\alpha_n = A^n\alpha_0$.*
(2) *Diagonalize A and compute A^n. Then find an explicit expression of F_n.*

General expressions of sequences with linear recursive relations can be computed using similar methods as above. Here is one more example.

Problem 5. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers which is determined by $a_0 = 0$ and*

$$a_{n+1} = \frac{1}{3 - 2a_n}.$$

Find the general term of a_n.

Use the method of the last problem. Don't guess an expression and prove it by induction.

Problem 6. *Consider the following two matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$*

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -3 & -2 & -3 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 2 \\ -3 & -6 & -3 \\ 2 & 2 & -1 \end{bmatrix}.$$

Determine whethere A, B can be simultaneously diagonalizable. If so, find a matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

Problem 7. *Given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$. Suppose that $I - A$ is invertible, where $I \in \mathrm{Mat}_{n\times n}(F)$ is the identity matrix. Show that $(I - A)^{-1}$ is a polynomial of A, i.e., there exists a polynomial $f \in F[x]$ such that $(I - A)^{-1} = f(A)$.*

Hint: If A is nilpotent, it should be easy to find such f. For the general A, consider the characteristic polynomial $\chi_A(x) = \det(xI - A)$. The assumption shows that $\chi_A(1) \neq 0$. Consider the long division $\chi_A(x) = (x - 1)p + r$ and use Cayley-Hamilton.