

HOMEWORK 3

Due date: Next Monday

Exercise: 4, 5, 7, 8 (page 21);

Exercise: 1, 3, 6, 7, 10 (page 27).

Note that Exercise 10, page 27 tells us that if $A \in \text{Mat}_{m \times n}(F)$ with $n < m$, then A has no right inverse in the following sense: there is no $B \in \text{Mat}_{n \times m}$ such that $AB = I_m$. Find an example of $A \in \text{Mat}_{m \times n}(F)$ with $n < m$ such that A has a left inverse, namely, such that there exists a matrix $C \in \text{Mat}_{n \times m}(F)$ with $CA = I_n$.

Problem 1. Let F be a field. For any $A \in \text{Mat}_{m \times n}(F)$ and $B \in \text{Mat}_{n \times m}(F)$, show that $\text{tr}(AB) = \text{tr}(BA)$.

Here recall that for a square matrix $C = (c_{ij})_{1 \leq i, j \leq n}$, $\text{tr}(C) = \sum_{i=1}^n c_{ii}$.

A square matrix $A \in \text{Mat}_{n \times n}(F)$ is called **nilpotent** if $A^k = 0$ for some integer $k > 0$. For example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is nilpotent, because $A^3 = 0$.

Problem 2. Let $B \in \text{Mat}_{n \times n}(F)$ be a nilpotent matrix. Show that $I_n - B$ is invertible.

There is no need to compute its inverse. Just show that the matrix is invertible.

Problem 3. Let F be a field, $A \in \text{Mat}_{m \times n}(F)$ and $B \in \text{Mat}_{n \times m}(F)$. Show that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

(Hint: Don't try to find the inverse matrix. Instead, use a criterion we learned in class to determine whether a matrix is invertible or not. The identity $A(I_n - BA) = (I_m - AB)A$ is helpful. This is Problem M.10, page 36 of Artin's book "Algebra", edition 2.)

Problem 4. Let F be a field, $A, B \in \text{Mat}_{n \times n}(F)$. If $AB = A + B$ show that $AB = BA$.

(Hint: consider $(A - I_n)(B - I_n)$.)

A matrix $A \in \text{Mat}_{n \times n}(F)$ is called a permutation matrix if each row and each column has only one nonzero entry and that non-zero entry is 1. For example,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is a permutation matrix. A matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$ is called upper triangular if $a_{ij} = 0$ for all i, j with $i > j$, namely, if each entry below the main diagonal is zero. Similarly, a matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$ is called lower triangular if $a_{ij} = 0$ for all i, j with $i < j$. For example

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is upper triangular. Here an entry with $*$ means that the value of that entry is not important. Moreover, it is easy to see that if a matrix $A \in \text{Mat}_{n \times n}(F)$ is a row reduced echelon matrix, then A is upper triangular. Following our book, an elementary row operation by interchanging two rows will be called **type III**. Similarly, one can define type III elementary matrix, which is just a matrix by applying a type III ERO to the identity matrix.

Problem 5. Determine what kind elementary matrices are upper triangular and what kind elementary matrices are lower triangular.

- Problem 6.**
- (1) Show that a matrix $A \in \text{Mat}_{n \times n}(F)$ is a permutation matrix if and only if it is a product of type III elementary matrices. What can you say about upper triangular matrices and lower triangular matrices using product of type I and type II elementary matrices?
 - (2) Show that each matrix $A \in \text{GL}_2(F)$ can be written as a product $A = LPU$, where $P \in \text{Mat}_{2 \times 2}(F)$ is a permutation matrix, $L \in \text{Mat}_{2 \times 2}(F)$ is lower triangular, and $U \in \text{Mat}_{2 \times 2}(F)$ is upper triangular. For what kind matrix $A \in \text{GL}_2(F)$, it has a decomposition $A = LU$?
 - (3) Show that each matrix $A \in \text{GL}_3(F)$ can be written as a product $A = LPU$, where $P \in \text{Mat}_{3 \times 3}(F)$ is a permutation matrix, $L \in \text{Mat}_{3 \times 3}(F)$ is lower triangular, and $U \in \text{Mat}_{3 \times 3}(F)$ is upper triangular.

(Actually, in part (3), there is no need to assume the matrix A is invertible. But the argument might be a little bit simpler when we add this condition.) Think about how to do part (3) for general $n \times n$ matrix. Namely, for any $A \in \text{GL}_n(F)$, show that it can be written as a product $A = LPU$, where $P \in \text{Mat}_{n \times n}(F)$ is a permutation matrix, $L \in \text{Mat}_{n \times n}(F)$ is a lower triangular matrix, and $U \in \text{Mat}_{n \times n}(F)$ is an upper triangular matrix. There is no need to submit a proof of this. But it is helpful to keep in mind this assertion. This decomposition of a matrix $A \in \text{GL}_n(F)$ is called the Bruhat decomposition.