HOMEWORK 4

NAME:

Due date: Monday

Exercises: 4, 7, (page 34)
Exercises: 9, (page 40)
Exercises: 7, 9, 11, 13 (page 49)

Problem 1. *Let F be a field and V be a vector space over F.*

1. *Show that the scaler product of the zero element of F with any vector in V is the zero vector.*
2. *Assume that $\dim_F V < \infty$. Let S be a finite subset which spans V, show that S contains a basis of V, namely, one can obtain a basis of V by deleting several elements from S.*
3. *Assume that $\dim_F V = n$ and $S = \{\alpha_1, \dots, \alpha_n\}$ is a set of n vectors. If S is linearly independent, show that S spans V and thus is a basis of V.*
4. *Assume that $\dim_F V = n$ and $S = \{\alpha_1, \dots, \alpha_n\}$ is a set of n vectors. If S spans V, show that S is linearly independent and thus is a basis of V.*

Problem 2. *Let F be a field and m, n be positive integers.*

1. *Show that $\dim_F \mathrm{Mat}_{m\times n}(F) = mn$.*
2. *For any $A \in \mathrm{M}_{n\times n}(F)$. Show that there is an integer N such that A satisfies a nontrivial polynomial equation*
$$A^N + c_{N-1}A^{N-1} + \cdots + c_1 A + c_0 I_n = 0.$$

(Hint: Consider the set $\left\{I_n, A, A^2, \dots, A^N\right\} \subset M_{n\times n}(F)$ for a suitable integer N. Then apply Theorem 4 of page 44 and part (1).)

Problem 3. *Let F be a field and let n be a positive integer. Let $V = \mathrm{Mat}_{n\times n}(F)$, which is a vector space of dimension n^2 over F by Problem 2. Let $W = \{A \in V \mid \mathrm{tr}(A) = 0\}$.*

1. *Show that W is a subspace of V;*
2. *Find a subspace U of V such that $U + W = V$.*
3. *Compute $\dim_F(W)$.*

Problem 4. *In this problem, you are assumed to know polynomials (of one variable/2-variables).*

1. *Let $x(t), y(t)$ be quadratic polynomials with real coefficients (for example, $x(t) = t^2 + t + 1, y(t) = t^2 - 2t - 3$). Show that there is a quadratic polynomial $f(x, y)$ (i.e., $f(x, y)$ is of the form $c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2$) such that $f(x(t), y(t))$ is identically zero.*
2. *Let $x(t) = t^2 - 1, y(t) = t^3 - t$. Find a nonzero real polynomial $f(x, y)$ such that $f(x(t), y(t))$ is identically zero.*
3. *Prove that every pair $x(t), y(t)$ of real polynomials satisfies some real polynomial relation $f(x, y) = 0$.*

(Hint: Use similar method as Problem 2).

Problem 5. *Let α be a real cubic root of 2 (i.e., $\alpha = \sqrt[3]{2}$).*

1. *Show that the set $\left\{1, \alpha, \alpha^2\right\}$ are linearly independent over $\mathbb{Q}$, i.e., if $a, b, c \in \mathbb{Q}$ such that $a + b\alpha + c\alpha^2 = 0$, then $a = b = c = 0$.*
2. *Show that the set $F = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathbb{Q}\}$ is a field.*

(Hint for (1): Proof by contradiction. Assume that $a, b, c \in \mathbb{Z}$ such that $a + b\alpha + c\alpha^2 = 0$ and divide $x^3 - 2$ by $cx^2 + bx + a$.)

Problem 6. *Set* $\alpha = \sqrt[3]{2}$. *Let* $F = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$. *We know that* F *is a field and has dimension 3 over* $\mathbb{Q}$ *from last problem. Thus* F^n *has dimension* $3n$ *as a* $\mathbb{Q}$ *vector space. Choose an ordered basis* $\mathcal{B}$ *of* F^n *(as a* $\mathbb{Q}$*-vector space). Compute* $[v]_{\mathcal{B}}$ *for* $v = (v_1, \ldots, v_n) \in F^n$, *where* $v_i = a_i + b_i\alpha + c_i\alpha^2$.

You **don't** have to do the next problem.

Problem 7. *Let* V *be a vector space over a field* F. *A subspace* W *of* V *is called proper if* $W \neq V$. *Show that if* F *is infinite,* V *is not the union of finitely many proper subspaces.*

(You can find a proof from this link. Please try to fill the details of the proof. You don't have to submit a solution of this problem.)