

HOMEWORK 6

Due date: Next Monday.

Exercises: 5, 11, 12, page 83-84

Exercises: 5, page 86

Exercises: 2, 10, 11, 12, page 95-96

Problem 1. Let V, W be two vector spaces over F and $T : V \rightarrow W$ be a linear transformation. Show that there is isomorphism

$$\bar{T} : V/\ker(T) \rightarrow \text{Im}(T)$$

defined by $\bar{T}(\alpha + \ker(T)) := T(\alpha)$.

The proof is given in class. Please repeat it here. Let $\mathbf{Vect}_F$ be the class of all vector spaces over F . An element of $\mathbf{Vect}_F$ is just a vector space over F .

Problem 2. Given $V_1, V_2, V_3 \in \mathbf{Vect}_F$ and $T_1 \in \text{Hom}_F(V_1, V_2), T_2 \in \text{Hom}_F(V_2, V_3)$. Show that

- (1) $\ker(T_1) \subset \ker(T_2 \circ T_1)$;
- (2) $T_1(\ker(T_2 \circ T_1)) \subset \ker(T_2)$.
- (3) There is an injective map $\tilde{T}_1 : \ker(T_2 \circ T_1)/\ker(T_1) \rightarrow \ker(T_2)$ defined by $\tilde{T}_1(\bar{x}) = T_1(x)$ for $x \in \ker(T_2 \circ T_1)$, where $\bar{x}$ represents the equivalence class of x , namely $\bar{x} = x + \ker(T_1)$. (You need to check that this map is well-defined, linear and injective. You can omit the “linear” part if you think it is easy.)
- (4) Assume that $\dim_F(V_i) < \infty$ for $i = 1, 2, 3$. Show that $\dim_F \ker(T_2 \circ T_1) \leq \dim_F \ker(T_1) + \dim_F \ker(T_2)$.
- (5) Given $A_2 \in \text{Mat}_{m \times n}(F)$ and $A_1 \in \text{Mat}_{n \times k}(F)$, show that

$$\text{rank}(A_1) + \text{rank}(A_2) - n \leq \text{rank}(A_2 A_1) \leq \min \{\text{rank}(A_1), \text{rank}(A_2)\}.$$

Hint: Consider the linear transformation $T_1 : F^k \rightarrow F^n, T_2 : F^n \rightarrow F^m$ defined by $T_i(X) = A_i X$, and use (4).

- (6) Given a matrix $A \in \text{Mat}_{m \times n}(F)$ with $\text{rank}(A) = r$ and a positive number p with $p < r$. Show that there does **not** exist matrices $C \in \text{Mat}_{m \times p}(F)$ and $R \in \text{Mat}_{p \times n}(F)$ such that $A = CR$. *Hint: This is a direct corollary of the last part.*

Comment: The inequality in (5) is called Sylvester’s rank inequality. Compare (6) with Problem 2 of HW 5. Do this problem step by step. It is not hard at all.

Problem 3. (1) Show that $\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$ as an $\mathbb{R}$ -vector space. Namely, there exists an $\mathbb{R}$ -linear isomorphism $T : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$.

- (2) Given $v = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = a_j + b_j \sqrt{-1} \in \mathbb{C}$ with $a_j, b_j \in \mathbb{R}$. Compute $T(v)$ for the isomorphism T you choose in part (1).

Problem 4. Consider the $\mathbb{R}$ -vector space $V = \mathbb{C}$. Then $\dim_{\mathbb{R}}(V) = 2$. Denote $e_1 = -1 + 7i$ and $e_2 = 5i$, where $i = \sqrt{-1} \in \mathbb{C}$.

- (1) Show that $\mathcal{B} = \{e_1, e_2\}$ is a basis of V over $\mathbb{R}$.
- (2) Let $z = x + yi \in \mathbb{C}$ with $x, y \in \mathbb{R}$. Consider the map $f_z : V \rightarrow V$ defined by $f_z(t) = zt$. Show that f_z is $\mathbb{R}$ -linear.
- (3) Let $T(z) \in \text{Mat}_{2 \times 2}(\mathbb{R})$ be the matrix of f_z with respect to the ordered basis $\mathcal{B}$. Show that

$$T(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}.$$

(4) Show that the map

$$T : \mathbb{C} \rightarrow \text{Mat}_{2 \times 2}(\mathbb{R})$$

$$z \mapsto T(z),$$

satisfies $T(z_1 z_2) = T(z_1)T(z_2)$ and $T(z_1 + z_2) = T(z_1) + T(z_2)$.

Note that the matrix $T(z)$ is the one given in Problem 5 of page 86 of the textbook. By Problem 5 of Page 86, the map $z \mapsto T(z)$ is injective, and satisfies $T(z_1 z_2) = T(z_1)T(z_2)$. These properties can be proved without explicit calculation, just using the fact that $T(z)$ is the matrix of the linear transformation f_z . The next problem is a “higher dimensional” version of Problem 5 of page 86.

Problem 5. Consider the $\mathbb{R}$ -vector space $V = \mathbb{C}^2$. Then $\dim_{\mathbb{R}}(V) = 4$.

- (1) Find an ordered basis $\mathcal{B}$ of V over $\mathbb{R}$.
- (2) For a matrix $A \in \text{Mat}_{2 \times 2}(\mathbb{C})$, consider the map $T_A : V \rightarrow V$ defined by $T_A(X) = AX$ for $X \in \mathbb{C}^2$, which is $\mathbb{R}$ -linear. Denote $R(A) = [T_A]_{\mathcal{B}}$, namely $R(A) \in \text{Mat}_{4 \times 4}(\mathbb{R})$ is the matrix of T_A with respect to $\mathcal{B}$ when V is viewed as a $\mathbb{R}$ -vector space. Show that the map $A \mapsto R(A)$ from $\text{Mat}_{2 \times 2}(\mathbb{C})$ to $\text{Mat}_{4 \times 4}(\mathbb{R})$ satisfies $R(A)R(B) = R(AB)$.
- (3) Compute $R(I_2)$, where $I_2 \in \text{Mat}_{2 \times 2}(\mathbb{C})$ is the identity matrix.
- (4) Show that if A is invertible, then $R(A)$ is also invertible. Thus we get a map $\text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_4(\mathbb{R})$.
- (5) Let $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Compute $R(A)$. (It depends on the ordered basis you chose in part (1)).

1. REAL VECTOR SPACE VS COMPLEX VECTOR SPACE

Let V be a vector space over $\mathbb{C}$. Then we can also view V as a vector space over $\mathbb{R}$ since $\mathbb{R}$ is a subfield of $\mathbb{C}$. Conversely, if we are given a vector space W over $\mathbb{R}$, in general, we might ask if it is possible to view W as a vector space over $\mathbb{C}$. There is an obvious restriction on $\dim_{\mathbb{R}} W$ if it is finite. What is it?

- Problem 6.** (1) Let V be a vector space over $\mathbb{C}$. Show that there is a linear map $J \in \text{End}_{\mathbb{R}}(V)$ such that $J \circ J = -I$.
- (2) Let W be a vector space over $\mathbb{R}$ and let $J \in \text{End}_{\mathbb{R}}(W)$ be a linear map such that $J \circ J = -I$. Show that we can define a $\mathbb{C}$ -vector space structure on W such that (1) The addition on W is the same; (2) The scalar product $\mathbb{C} \times W \rightarrow W$ is defined by $(a + b\sqrt{-1}) \cdot v = av + bJ(v)$, $a, b \in \mathbb{R}$.

If $\dim_{\mathbb{R}}(W)$ is finite, a necessary condition for the existence of linear map $J \in \text{End}_{\mathbb{R}}(W)$ such that $J \circ J = -I$ is that $\dim_{\mathbb{R}}(W)$ must be even.

2. COMPLEX LINEAR VS REAL LINEAR

Let V be a vector space over $\mathbb{C}$. We can also view V as a vector space over $\mathbb{R}$ since $\mathbb{R} \subset \mathbb{C}$. Let $T \in \text{End}_{\mathbb{R}}(V)$ be an $\mathbb{R}$ -linear map, which means $T(av) = aT(v)$ for all $a \in \mathbb{R}, v \in V$. In general, T is not $\mathbb{C}$ -linear. A linear map $T \in \text{Hom}_{\mathbb{R}}(V, V)$ is called conjugate linear if $T(zv) = \bar{z}T(v)$, for all $z \in \mathbb{C}, v \in V$.

Problem 7. Let $V = \mathbb{C}$ be the 1-dimensional vector space over $\mathbb{C}$. Consider the $\mathbb{R}$ -linear map $T : V \rightarrow V$ given by

$$T(a + ib) = 2a + b + i(a + 3b),$$

where $i = \sqrt{-1}$. Determine whether T is $\mathbb{C}$ -linear, conjugate linear or neither.

Problem 8. Let V be a vector space over $\mathbb{C}$. Consider the linear map $J : V \rightarrow V$ defined by $J(v) = \sqrt{-1}v$ for $v \in V$. Note that $\sqrt{-1}v$ makes sense because V is a vector space over $\mathbb{C}$.

- (1) Show that T is $\mathbb{C}$ -linear iff $TJ = JT$. Find a similar condition for the conjugate linear map T .
- (2) For any $T \in \text{End}_{\mathbb{R}}(V)$, show that there is a unique $\mathbb{C}$ -linear map $T_1 \in \text{End}_{\mathbb{C}}(V)$ and a conjugate linear map $T_2 \in \text{End}_{\mathbb{R}}(V)$ such that $T = T_1 + T_2$. Find such T_1, T_2 for the linear map T in Problem 7.