HOMEWORK 7

Due date: Next Monday.
Exercises: 1, 8, 9, 12, pages 105-107
Exercises: 1, 3, page 111
Exercises: 1, 3, 4, 6, 7, 8, pages 115-116.

You can do Problems 1, 2 and 3 using what you learned from last week.

Problem 1. *Given two matrices $A, B \in \mathrm{Mat}_{m\times n}(F)$. Show that* $\mathrm{rank}(A+B) \leq \mathrm{rank}(A)+\mathrm{rank}(B)$.

Hint: Translate this problem to a problem on linear maps. You might find the following fact useful: Given two linear maps $T, S : V \to W$, then $\ker(T) \cap \ker(S) \subset \ker(T+S)$ (check this).

Problem 2. *Given two matrices $A, B \in \mathrm{Mat}_{n\times n}(F)$. If $AB = 0$, show that* $\mathrm{rank}(A)+\mathrm{rank}(B) \leq n$.

This follows from Sylverster's rank inequality (Problem 3, HW6). But it is a good exercise to prove this directly (without going through the whole proof of Sylverster, because this problem is much easier.) Hint: Think about what can you say about $\mathrm{Im}(T_B)$ and $\mathrm{Ker}(T_A)$. (As usual: T_A denotes the linear map $F^n \to F^n$ given by $T_A(X) = AX$. Here elements in F^n are viewed as column vectors.)

Problem 3. *Let F be a fixed field and n be a positive integer. Denote by $I \in \mathrm{Mat}_{n\times n}(F)$ the identity $n \times n$ matrix. Given $A, B \in \mathrm{Mat}_{n\times n}(F)$.*

(1) *Show that*
$$\mathrm{rank}(A - ABA) = \mathrm{rank}(A) + \mathrm{rank}(I - BA) - n.$$
(2) *Show that*
$$\mathrm{rank}(I - AB) = \mathrm{rank}(I - BA).$$

Hint: The equation $A - ABA = A(I - BA) = (I - AB)A$ might be useful.

Problem 4. *Denote $\alpha = \sqrt[3]{2}$. Consider $F = \{a + b\alpha + c\alpha^2 | a, b, c \in \mathbb{Q}\}$. We know that F is a field and it is also a vector space over $\mathbb{Q}$ of dimension 3 from previous HW. We view it as a $\mathbb{Q}$-vector space.*

(1) *Given $x = a + b\alpha + c\alpha^2 \in F$ and the linear map $T_x : F \to F$ given by $T_x(y) = xy$. It is not hard to see T_x is well-defined and $\mathbb{Q}$-linear. Here "well-defined" means $T_x(y) \in F$ for $y \in F$. Suppose that $x \neq 0$. Show that T_x is injective and conclude that there exists a $y \in F$ such that $xy = 1$.*
(2) *Fix an ordered basis $\mathcal{B}$ of F (as a $\mathbb{Q}$ vector space) and compute the matrix $[T_x]_{\mathcal{B}}$ of T_x with respect to the basis you chose.*
(3) *Show that $[T_x]_{\mathcal{B}}$ is invertible.*
(4) *Do a higher dimensional analogue of this. For example, given a matrix $A \in \mathrm{Mat}_{2\times 2}(F)$, and consider the linear map $T_A : F^2 \to F^2$. View F^2 as a $\mathbb{Q}$-vector space and choose a basis $\mathcal{B}'$ of F^2 over $\mathbb{Q}$. Then compute $[T_A]_{\mathcal{B}'} \in \mathrm{Mat}_{6\times 6}(\mathbb{Q})$ explicitly in terms of entries of A. Show that if A is invertible as an element of $\mathrm{Mat}_{2\times 2}(F)$ then $[T_A]_{\mathcal{B}}$ is invertible as an element of $\mathrm{Mat}_{6\times 6}(\mathbb{Q})$.*

Exercise 16 page 107 gives a "coordinate free" definition of trace of a square matrix. Please keep in mind this assertion. In Chapter 5, we will see a "coordinate free" definition of determinant.

Problem 5. *Let F be a field and let $V = \mathrm{Mat}_{n\times n}(F)$, which is an F-vector space of dimension n^2. We consider the trace map $\mathrm{Tr} : V \to F$. Let W be the subspace of V which is spanned by the matrices of the form $AB - BA$ for $A, B \in V$. Then we know that $W \subset \ker(\mathrm{Tr})$. This space W is exactly the one in Ex 17, page 107.*

(1) *Show that* $\mathrm{Tr} : V \to F$ *is surjective and conclude that* $\dim \ker(\mathrm{Tr}) = n^2 - 1$.
(2) *Show that* $\dim W = n^2 - 1$ *by explicitly constructing enough linearly independent elements in* W. *Conclude that* $W = \ker(\mathrm{Tr})$.
(3) *Show that a linear functional* $f : V \to F$ *such that* $f(AB) = f(BA)$ *for all* $A, B \in F$ *is exactly an element in* $\mathrm{Hom}_F(V/W, F)$. *Conclude that such an* f *must be of the form* $c\mathrm{Tr}$ *for some* $c \in F$.

Hint for part (2). We expect that $W = \mathrm{Ker}(\mathrm{tr})$. Try to find a natural basis of $\ker(\mathrm{tr})$ (if you still have no idea, try to think about the 2×2 matrices and then 3×3 matrices), and then try to show that they are indeed in W.

Problem 6. *Given* $V, W \in \mathrm{Vect}_F$ *such that* $\dim V, \dim W$ *are finite. Let* $T : W \to V$ *be a linear operator.*

(1) *Given a linear functional* $f \in W^*$ *such that* $f|_{\ker(T)} = 0$, *show that there exists a linear functional* $g \in V^*$ *such that* $g(T(\alpha)) = f(\alpha)$ *for any* $\alpha \in W$.
(2) *If* T *is injective, conclude that* T^t *is surjective.*

Part (1) is a variant/generalization of Ex 12, page 106.

Problem 7. *Let* V, W *be two finite dimensional vector spaces over* F. *Suppose* $\dim_F V = n, \dim_F W = m$.

(1) *Show that the map* $\theta : \mathrm{Hom}_F(V, W) \to \mathrm{Hom}_F(W^*, V^*)$ *defined by* $\theta(T) = T^t$ *is an isomorphism.*
(2) *Conclude that there is an isomorphism* $\mathrm{Hom}(V, W) \to (V^*)^m$. *Construct this isomorphism explicitly.*

(Comment: Part (1) is a generalization of Ex. 7, page 116.)

We can prove the assertion in part (2) of the above problem even we drop the condition that $\dim V$ is finite.

Problem 8. *Given two* F*-vector spaces* V, W *with* $\dim_F W = m$. *We don't require* $\dim_F V$ *is finite. Let* $\{\beta_1, \ldots, \beta_m\}$ *be a basis of* W *and let* $S = \{f_1, \ldots, f_m\}$ *be the dual basis of* W^*. *Consider the map*

$$\theta_S : \mathrm{Hom}(V, W) \to (V^*)^m,$$
$$\theta_S(T) = (T^t(f_1), \ldots, T^t(f_m)).$$

Show that θ_S *is an isomorphism.*

(Hint: The proof is not hard.) The above assertion slightly generalizes Ex.6 page 105. Actually Exercise 6 of page 105 gives an inverse map of the one defined above (for a specific choice of S). Try to explain this. Moreover, compare the result in Problem 6 with Problem 1 of HW6.

Let V be a vector space over a field F such that $\dim_F V = n$ is finite. We know that V is isomorphic to V^* since both of them have the same dimension. But this isomorphism is not "canonical". On the other hand, the isomorphism $V \to V^{**}$ given by $\alpha \mapsto L_\alpha$ is "canonical". The next problem is trying to give you a feeling what the word "canonical" means here.

Problem 9. *Let* V *be a finite dimensional vector space and consider the map*

$$\Theta_V : V \to V^{**}$$

defined by $\Theta_V(\alpha) = L_\alpha$, *where* $L_\alpha : V^* \to F$ *is defined by* $L_\alpha(f) = f(\alpha), f \in V^*$. *Let* $T : V \to W$ *be a linear map such that both* V *and* W *are finite dimensional. Show that the following diagram is commutative*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Theta_V"] & W \arrow[d, "\Theta_W"] \\ V^{**} \arrow[r, "(T^t)^t"] & W^{**}. \end{tikzcd}$$

Namely, show that $(T^t)^t \circ \Theta_V = \Theta_W \circ T$.

Problem 10. *Let F be a field and consider the vector space $V = F^n$ for a positive integer n. An element of V is viewed as a row vector. Let m, k be positive integers.*

(1) *For $\alpha = (x_1, \ldots, x_n) \in V$, define $f_\alpha \in V^*$ by $f_\alpha((y_1, \ldots, y_n)) = x_1 y_1 + \cdots + x_n y_n$. The map $f : V \to V^*, \alpha \mapsto f_\alpha$ is known to be linear. Show that f is an isomorphism. (You don't have to check that $f_\alpha \in V^*$ and $\alpha \mapsto f_\alpha$ is linear).*

(2) *Let $A \in \mathrm{Mat}_{m\times n}(F)$ and let $\mathrm{Row}(A) \subset V$ be the row space of A (subspace of V spanned by rows of A). Show that $\alpha \in \mathrm{Ker}(A)$ if and only if $f_\alpha \in \mathrm{Ann}(\mathrm{Row}(A))$. Here $\mathrm{Ker}(A) = \{\alpha \in V : A\alpha = 0.\}$ (In the equation $A\alpha = 0$, α is identified with a column vector).*

(3) *Let $A \in \mathrm{Mat}_{m\times n}(F)$ and $B \in \mathrm{Mat}_{k\times n}(F)$ be two matrices. Show that $\mathrm{Ker}(A) = \mathrm{Ker}(B)$ if and only if $\mathrm{Row}(A) = \mathrm{Row}(B)$.*

Note that part (3) shows that two homogeneous linear systems $AX = 0$ and $BX = 0$ have the same solutions iff they are equivalent. Here "equivalence" is defined in page 4 of the textbook.