

HOMEWORK 8

Due date: Monday

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Let $\mathbb{F}_p$ be the field with p -elements, where p is a prime number. Recall that this field is constructed using equivalence classes. A different way to write this field is $\mathbb{Z}/p$.

Let F be a fixed field. Let $V_n = V_n(F)$ be the F -vector space of F -polynomials of degree $\leq n$. Then $\dim_F V_n = n + 1$ and thus $\dim_F V_n^* = n + 1$. Given $t \in F$, we have defined $L_t \in V_n^*$ by $L_t(f) = f(t)$ for $f \in V_n$. Lagrange interpolation says that if $t_0, \dots, t_n$ are distinct points in F , then $\{L_{t_i} : 0 \leq i \leq n\}$ is a basis of V_n^* .

Problem 1. Let $F = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ be the field of 5 elements. Consider $V_3^*(F)$ which has dimension 4 and thus $L_0, L_1, L_2, L_3, L_4 \in V_3^*$ are linearly dependent. Write L_4 as a linear combination of L_0, L_1, L_2, L_3 .

Problem 2. Let S be any subset of F . Show that the subset $\{L_s : s \in S\} \subset \text{Hom}_F(F[x], F)$ is linearly independent where L_s is viewed as a linear function on $F[x]$ defined by $L_s(f) = f(s)$, $f \in F[x]$.

The formal power series algebra $F[[x]]$ is written as F^∞ in the book. We also used the notation $F^\mathbb{N}$ to denote $F[[x]]$.

Problem 3. Given a formal power series $f = (f_0, f_1, \dots, f_n, \dots) \in F^\mathbb{N}$, we consider the map $\phi_f : F[x] \rightarrow F$ defined by

$$\phi_f(x^i) = f_i$$

or

$$\phi_f(a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = a_0f_0 + a_1f_1 + \dots + a_nf_n.$$

- (1) Show that ϕ_f is linear and thus $\phi_f \in \text{Hom}_F(F[x], F)$.
- (2) Show that the map $\phi : F[[x]] \rightarrow \text{Hom}_F(F[x], F)$ is F -linear.
- (3) Show that ϕ is an isomorphism by explicitly constructing an inverse of ϕ .
- (4) Given an element $t \in F$. By part (3), we know that L_t must be of the form ϕ_f for some $f \in F[[x]]$. Describe f in terms of t .

(Comment: Even we cannot compare dimensions because both $F[x]$ and $F[[x]]$ are infinite dimensional as F -vector spaces, it should be clear that $F[[x]]$ is strictly larger than $F[x]$. You might find a proof of this online. But we won't show it in this course. This problem shows that the dual of $F[x]$ is strictly larger than $F[x]$, which never happens in the finite dimension case.

You don't have to do the next problem. But try to think about it.

Problem 4. What is the dual of $F[[x]]$? Is it the same as $F[x]$? Define a map $F[x] \rightarrow \text{Hom}_F(F[[x]], F)$ and show it is injective. Determine if it is surjective or not. If it is, prove it. If it is not, give an element in $\text{Hom}_F(F[[x]], F)$ which is not in the image.

- Problem 5.**
- (1) Find a nonzero polynomial $f \in \mathbb{F}_p[x]$ such that $f(a) = 0$ for any $a \in \mathbb{F}_p$.
 - (2) Let $f \in \mathbb{F}_p[x]$ be a nonzero polynomial such that $f(a) = 0$ for any $a \in \mathbb{F}_p$. Show that $\deg(f) \geq p$.
 - (3) Consider the set $I = \{f \in \mathbb{F}_p[x] : f(a) = 0, \forall a \in \mathbb{F}_p\}$. Show that I is ideal of $\mathbb{F}_p[x]$.

Hint: (2) is a consequence of Lagrange interpolation. Of course, you can also compare the number of roots of a polynomial and the degree of the given polynomial.

Problem 6. Let $A \in \text{Mat}_{n \times n}(F)$ be a fixed non-zero polynomial. Consider the set

$$I = \{f \in F[x] : f(A) = 0\}.$$

Show that I is a (nonzero) ideal. Suppose that d is the nonzero monic polynomial such that $I = dF[x]$. Show that $\deg(d) \leq n^2$.

1. DERIVATIVES

Derivatives defined in your calculus class can be defined in the following way. Let $U \subset \mathbb{R}$ be an open subset set (if you don't know what an open subset means, just view it as an open interval) and let $f : U \rightarrow \mathbb{R}$ be a map. Given a point $a \in U$, we say that f is differentiable at a if there is an $\mathbb{R}$ -linear map $T : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(1.1) \quad \lim_{h \rightarrow 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0.$$

Here the absolute value $|\cdot|$ is the usual one in $\mathbb{R}$.

Problem 7. Given a function $f : U \rightarrow \mathbb{R}$ and a point $a \in U$. Suppose that f is differentiable at a . In other words, there is a linear map T such that (1.1) holds.

- (1) Show that the linear map $T : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1.1) is unique. It is called the derivative of f at a . To emphasize the dependence on a and f , we write it as $df|_a$.
- (2) Show that the definition of $df|_a$ is the same as what you learned in calculus class. More precisely, the linear map $df|_a : \mathbb{R} \rightarrow \mathbb{R}$ is given by $df|_a(x) = f'(a)x$.

The above definition can be generalized easily. Let n, m be two possible integers. Let $U \subset \mathbb{R}^n$ be an open subset (if you don't know what an open set is, just take $U = \mathbb{R}^n$) and let $F : U \rightarrow \mathbb{R}^m$. Given a point $a \in U$. The function F is called differentiable at a if there exists an $\mathbb{R}$ -linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$(1.2) \quad \lim_{h \rightarrow 0} \frac{|F(a+h) - F(a) - T(h)|}{|h|} = 0.$$

Some explanations. Here $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ and $|h| = \sqrt{h_1^2 + \dots + h_n^2}$. Note that $F(a+h) - F(a) - T(h)$ takes value in $\mathbb{R}^m$. For a point $y = (y_1, \dots, y_m)$, its absolute value $|y|$ is defined by $|y| = \sqrt{y_1^2 + \dots + y_m^2}$.

Problem 8. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function such that it is differentiable at a . Namely, there exists a linear map T such that (1.2) holds. Show that the linear map T satisfying (1.2) is unique. To emphasize the dependence on F and a , we write it as $D(F)|_a$. The linear map $D(F)|_a$ is called the derivative of F at a .

If we use the standard basis $\mathcal{B}$ (resp. $\mathcal{B}'$) of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$), the linear map $D(F)|_a$ is given by a matrix $[D(F)|_a]_{\mathcal{B}, \mathcal{B}'} \in \text{Mat}_{m \times n}(\mathbb{R})$. Usually, we always use $\mathcal{B}$ and $\mathcal{B}'$ as the basis and don't emphasize them. Thus we usually identify the linear map $D(F)|_a$ and the matrix $[D(F)|_a]_{\mathcal{B}, \mathcal{B}'}$ that represents it. Namely, $D(F)|_a$ also denotes the matrix of this linear map. With this convention, $T(h)$ in (1.2) can be interpreted as a matrix multiplication $D(F)|_a h$, where $D(F)|_a \in \text{Mat}_{m \times n}(F)$ and $h \in \text{Mat}_{n \times 1}(F)$ is viewed as a column vector.

The following problem gives several simple examples.

Problem 9. In this problem, take $n = 2, m = 1$.

- (1) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = xy$ and the point $a = (x_0, y_0) \in \mathbb{R}^2$ for any fixed $x_0, y_0 \in \mathbb{R}$. Show that f is differentiable at a and $D(f)|_a = (y_0, x_0) \in \text{Mat}_{1 \times 2}(\mathbb{R})$.
- (2) Let $\alpha \in \mathbb{R}$. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2)^\alpha$. Determine for what values of α , the function f is differentiable at $a = (0, 0)$.