HOMEWORK 1

Due date: Monday of Week 2
Exercises: 1, 2, 5, 9, 10, page 213
Exercises: 3, 5, pages 218-219
Exercises: 2, 5, 6, 13, 14, 15. pages 225-226

For Problem 6 page 226, give a condition such that T is diagonalizable and give a condition such that T is nilpotent. Hint: We had a problem before to tell you something about rank 1 matrices.

In the following, F is a general field. If it is necessary, you can assume that the characteristic of F is zero.

Let V be a finite dimensional vector space over a field F and let $T \in \mathrm{End}_F(V)$. Let W be a T-invariant subspace of V with $W \neq 0, W \neq V$. We can consider the following question. Is there always a T-invariant subspace W' of V such that $V = W \oplus W'$. The following is an example.

Problem 1. *Let V be a vector space over F with $\dim_F V = 2$. Let $\mathcal{B} = \mathrm{Span}\{\alpha_1, \alpha_2\}$ be a basis of V. Consider*

$$T : V \to V$$

defined by $T(\alpha_1) = \alpha_1, T(\alpha_2) = a\alpha_1 + \alpha_2$ for some $a \in F$. Let $W = \mathrm{Span}\{\alpha_1\}$. Then W is T invariant. Determine whether there exists a $W' \subset V$ such that W' is T-invariant and $V = W \oplus W'$.

This problem is similar to Exercise 2, page 218. We will go back to this problem in Section 7.5.

Problem 2. *Let $\rho : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$ be a map such that $\rho(z_1 z_2) = \rho(z_1)\rho(z_2), \forall z_1, z_2 \in \mathbb{C}^\times$.*

(1) *Construct such a ρ such that $\rho(z)$ is not diagonalizable for any $|z| \neq 1$;*
(2) *Suppose that there is an element z_0 such that $\rho(z_0)$ has two distinct eigenvalues. Show that there exists a matrix $P \in \mathrm{GL}_2(\mathbb{C})$ such that*

$$P^{-1}\rho(z)P$$

is diagonal for any $z \in \mathbb{C}^\times$.

Hint for (1): think about what kind matrices in $\mathrm{GL}_2(\mathbb{C})$ are not diagonalizable.

Problem 3. *Consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ with p-elements for a prime p. Consider $\mathbb{F}_p^\times = \{x \in \mathbb{F}_p : x \neq 0\}$. It is known that for any element $x \in \mathbb{F}_p^\times$, we have $x^{p-1} = 1$. This is called Fermat's little theorem. We will show this later (or you can try to prove this on your own). Let $\rho : \mathbb{F}_p^\times \to \mathrm{GL}_n(\mathbb{C})$ be a map such that $\rho(x_1 x_2) = \rho(x_1)\rho(x_2)$ for any $x_1, x_2 \in \mathbb{F}_p^\times$.*

(1) *Show that $\rho(1) = I_n$.*
(2) *Show that there exists an element $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P\rho(x)P^{-1}$ is diagonal for any $x \in \mathbb{F}_p^\times$.*

Problem 4. *Let $W_i, 1 \leq i \leq k$ be subspaces of V. Let $\iota_i : W_i \to W$ be the linear map defined by $\iota_i(\alpha_i) = \alpha_i$. This makes sense because W_i is a subspace of V.*

(1) *Suppose*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

Given any vector space X and any linear map $f_i : W_i \to X$, show that there is a unique linear map $f : V \to X$ such that $f_i = f \circ \iota_i$ for each i with $1 \leq i \leq k$. In other words, there exists a commutative diagram

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

(2) *Given any vector space X and any linear map $f_i : W_i \to X$, suppose there is a unique linear map $f : V \to X$ such that $f_i = f \circ \iota_i$ for each i with $1 \le i \le k$. In other words, there exists a commutative diagram*

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

show that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

Suppose that F is a field of characteristic zero. The next problem is similar to Exercise 2, page 225.

Problem 5. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}.$$

(1) *Show that A is not diagonalizable.*
(2) *Find the Jordan decomposition of A, namely, find a diagonalizable matrix D and a nilpotent matrix N such that $A = D + N$ and $DN = ND$.*

The primary decomposition theorem can be proved inductively as follows.

Problem 6. *Let V be a vector space over a field F and let $T : V \to V$ be a linear operator.*

(1) *Let $p, q \in F[x]$ be two relatively prime polynomials. Show that $\ker(p(T)q(T)) = \ker(p(T)) \oplus \ker(q(T))$. Here we don't have to assume that $\dim V$ is finite.*
(2) *Deduce the primary decomposition theorem from part (1). Namely, if $\dim V$ is finite and the minimal polynomial μ_T is written as*

$$\mu_T = p_1^{r_1} \dots p_k^{r_k}$$

where the p_i are distinct irreducible monic polynomials in $F[x]$ and r_i are positive integers, then

$$V = W_1 \oplus \cdots \oplus W_k,$$

with $W_i = \ker(p_i(T)^{r_i})$.

The following is an application of primary decompositions.

Problem 7. *Consider the Fibonacci sequence $\{F_n\}_{n \ge 0}, F_n \in \mathbb{C}$, which is determined by $F_0 = F_1 = 1$ and*

$$F_{n+2} = F_{n+1} + F_n, n \ge 0.$$

Consider the vector space $V = \mathbb{C}[[x]]$, the formal power series ring over $\mathbb{C}$. Here we only view $\mathbb{C}[[x]]$ as a vector space over $\mathbb{C}$. Recall that each element of V is an infinite sequence of the form

$$(a_0, a_1, a_2, \dots, a_n, \dots), a_j \in \mathbb{C}.$$

We consider the linear operator $T : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by

$$T(a_0, a_1, a_2, \dots, a_n, \dots) = (a_1, a_2, \dots, a_n, a_{n+1}, \dots).$$

Now view the Fibonacci sequence

$$\mathbb{F} = (F_0, F_1, \dots, F_n, \dots)$$

as an element of $\mathbb{C}[[x]]$.

(1) *Show that $(T^2 - T - I)(\mathbb{F}) = 0$.*
(2) *Let λ_1, λ_2 be the two roots of $x^2 - x - 1 = 0$. Show that there exists $a_0, b_0 \in \mathbb{C}$ such that $F_n = a_0 \lambda_1^n + b_0 \lambda_2^n$ for all $n \ge 0$. Here a_0, b_0 could be computed using the equations when $n = 0, 1$.*

For part (2), we have $\mathbb{F} = \mathrm{Ker}(T^2 - T - I)$, which is $\mathrm{Ker}(T - \lambda_1 I) \oplus \mathrm{Ker}(T - \lambda_2 I)$ by primary decomposition, part (1) of Problem 6.

1. Additional problems on primary decomposition

You don't have to submit solutions of the rest problems. But it is worth to think about them. These problems should be with you when you read the book. But I don't have time to address these questions in classes. Again, try to ask yourself reasonable questions when you read math books. These problems are related to Exercise 4, page 225. Given a $T \in \mathrm{End}_F(V)$.

Problem 8. *In the primary decomposition theorem, if* $\mu_T = p_1^{r_1} \dots p_k^{r_k}$, *we have*

$$V = W_1 \oplus \cdots \oplus W_k,$$

with $W_i = \ker(p_i(T)^{r_i})$. *What can you say about* $\dim W_i$? *Show that* $W_i \neq 0$ *at least.*

Problem 9. *Let* $f \in I(T) = \{g \in F[x] : g(T) = 0\}$ *be a nonzero polynomial and let*

$$f = p_1^{s_1} \dots p_t^{s_t}$$

be the prime decomposition of f *with distinct irreducible polynomials* $p_1, \dots, p_k$ *and* $s_i \geq 0$. *Let* $W_i' = \ker(p_i(T)^{s_i})$. *Show that*

1. $V = W_1' \oplus W_2' \oplus \cdots \oplus W_t'$;
2. *each* W_i' *is* T*-invariant.*
3. *If* $p_j \nmid \mu_T$, *show that* $W_j' = \{0\}$.
4. *If* $p_i | \mu_T$ *and* $p_i | f$, *show that* $W_i = W_i'$, *namely,* $\ker(p_i(T)^{r_i}) = \ker(p_i(T)^{s_i})$.
5. *From the last two parts, apparently, we cannot expect* $p_i^{s_i}$ *is the minimal polynomial of* $T|_{W_i'}$.

Problem 10. *Do we know that* μ_T *and* χ_T *have exactly the same prime factors? Is it possible that there exists a prime polynomial such that* $p | \chi_T$ *but* $p \nmid \mu_T$*?*

The answer is no. See section 7.2. But it is a good exercise to think about this by yourself. We proved that μ_T and χ_T have the same roots. But it is possible that over a field, a prime factor p does not have any root. So if there exists a prime factor $p | \chi_T$ but $p \nmid \mu_T$, it does not contradict to what we know so far. But why cannot this happen?

Now suppose that

$$\chi_T = p_1^{d_1} \dots p_k^{d_k}.$$

Now let $W_i' = \ker(p_i(T)^{d_i})$ for $1 \leq i \leq k$. The above problems show that $W_i' = W_i$ and the primary decomposition obtained using χ_T and the primary decomposition obtained using $f = \mu_T$ are exactly the same.

Now think about the following question:

Problem 11. *Why do we use* μ_T *rather than* χ_T *in the statement of the primary decomposition theorem even it gives the same decomposition when we replace* μ_T *by* χ_T*?*

To help you to understand the above decompositions, try to work out the following example.

Problem 12. *Let* $V = F^4$ *and* $T : V \to V$ *is represented by*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have $\mu_T = x^2(x-1)$ *and* $\chi_T = x^2(x-1)^2$. *Also consider the polynomial* $g = x^2(x-1)(x^2+1) \in I(T)$. *Compute* $W_1 = \ker(T^2), W_2 = \ker(T-I)$; *and* $W_1' = \ker(T^2)$ *and* $W_2' = \ker(T-I)^2$; *and* $\ker(T^2+I)$. *Here we assume that the characteristic of* F *is not* 2.