HOMEWORK 2

Due date: Next Monday

The main task of this HW is to finish all exercises in Section 7.2. I will talk some of them in class.

Exercises 4, 7, 8, page 231.

Do all of Exercises of section 7.2. If you think there are too many, at least try to do the following problems. Exercises: 2, 3, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21. Pages 241-244.

For Ex.8, $\mu_T = (x-1)^2, \chi_T = (x-1)^3$.

For Ex 9, $\chi_T = (x+2)^2(x-1)$, $\mu_T = (x+2)(x-1)$.

Ex 12 was one general exam (an oral exam) problem of Princeton University (Every Ph.D candidate should pass the oral exam,) see this link. This Question is more general: **How do you determine if two matrices are similar or not?** (answer: by computing its invariant factors) **What theorem are you using?** (answer: cyclic decomposition theorem, or structure theorem of modules over PID; the former is a special case of the latter) **Why is $k[T]$ is PID?** (Here k is a field, $k[T]$ is the polynomial ring over k, and PID stands for "principal ideal domain". It is clear that $k[T]$ is a domain, or integral domain (an integral domain is a commutative ring such that if $ab = 0$ then $a = 0$ or $b = 0$). This question asks you how to prove that every ideal of $k[T]$ is principal. (answer: using Euclidean division.) **If two matrices are conjugate over the algebraic closure of the base field, does that mean the two matrices are conjugate over the base field?** (The last part is roughly Ex 12. "conjugate" is a different way to say "similar" in this situation. As you can see, the answer is Yes.)

You can also check https://web.math.princeton.edu/generals/ for general exams of some famous Princeton math graduated students, and possibly try to prepare them when you learn something relevant new materials. Hopefully some of you will have the chance to take and pass this exam.

Problem 1. *Let K be an algebraically closed field and $F \subset K$ be a subfield. Assume that the characteristic of F (and hence of K) is zero.*

(1) *Let $f \in F[x]$ be an irreducible polynomial. Show that f has no repeated root in K, which means for any root $c \in K$ of f, $(x-c)^2$ does not divide f.*
(2) *Let $A \in \mathrm{Mat}_{n\times n}(F) \subset \mathrm{Mat}_{n\times n}(K)$. Let $T : F^n \to F^n$ be the F-linear map defined by the matrix A and let $U : K^n \to K^n$ be the K-linear map defined by the same matrix A. Suppose that the only subspaces of T are F^n and the zero subspace, show that U is diagonalizable.*

This is roughly the same problem as Exercise 21.

Here, "F is a subfield of K" means that F is a subset of K and it is closed under the addition and multiplication of K such that F itself is also a field together with these two operations. Examples: $F = \mathbb{Q}, K = \mathbb{C}$; $F = \mathbb{R}, K = \mathbb{C}$; $F = \mathbb{Q}[\sqrt[3]{2}], K = \mathbb{C}$.

Problem 2. *Let n be a positive integer and $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ satisfying $A^2 + B^2 = AB$. If $AB - BA$ is invertible, show that n is a multiple of 3.*

For a solution, see this link.