HOMEWORK 3

Due date: Monday of Week 4
Exercises: 1, 2, 3, 6, 8, 10, 11, 12, 13, 15, 16, pages 250-251 of Hoffman-Kunze,

The next problem is not closely related to the materials of this week.

Problem 1. *Given two matrices* $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

(1) *Show that* $\chi_{AB} = \chi_{BA}$.
(2) *Suppose that* $\deg(\mu_{AB}) > \deg(\mu_{BA})$, *show that* $\mu_{AB} = x\mu_{BA}$. *Here* $x\mu_{BA}$ *denote the product of the polynomial* x *with the minimal polynomial* μ_{BA} *of* BA.
(3) *Suppose that* AB *is diagonalizable, show that* $(BA)^2$ *is diagonalizable.*
(4) *Give one example such that* AB *is diagonalizable but* BA *is not diagonalizable.*
(5) *Let* $\lambda \in \mathbb{C}, \lambda \neq 0$ *and* $r \in \mathbb{Z}$ *be a positive integer. Show that* $\dim \mathrm{Ker}(\lambda I - AB)^r = \dim \mathrm{Ker}(\lambda I - BA)^r$.

The above problem was borrowed from this link.

Given a positive integer n, a partition λ of n is a sequence of decreasing positive numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. We write $\lambda \vdash n$. Given a sequence of decreasing positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$, we also write $|\lambda| = \sum_{i=1}^k \lambda_i$. Thus $\lambda \vdash |\lambda|$. For example $(2,2) \vdash 4, (2,1,1) \vdash 4$. Given a positive integer n, let $\mathcal{P}(n)$ be the set of all partitions of n. Let $P(n) = |\mathcal{P}(n)|$, which is the number of all partitions of n. For example,

$$\mathcal{P}(4) = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}.$$

Thus $P(4) = 5$.

Problem 2. (1) *Compute* $\mathcal{P}(n)$ *and* $P(n)$ *for* $n = 5, 6$.
(2) *Show that* $P(n)$ *is the coefficient of* x^n *in the formal power series*

$$\prod_{m=1}^{\infty} \left(\frac{1}{1-x^m}\right)$$
$$=(1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots)\dots$$

Recall that a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is called nilpotent if $A^k = 0$ for some $k > 0$. We denote $\mathfrak{n}_n(\mathbb{C})$ the subset of nilpotent matrices in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. On $\mathfrak{n}_n(\mathbb{C})$, we define an equivalence relation R by similarity, namely, $R = \{(A, B) \in \mathfrak{n}_n(\mathbb{C}) \times \mathfrak{n}_n(\mathbb{C}) : A \text{ is similar with } B\}$. We consider the equivalence class $\mathfrak{n}_n(\mathbb{C})/R$. An element of $\mathfrak{n}_n(\mathbb{C})/R$ (which is an equivalence class) is called a conjugacy class of a nilpotent matrix. Recall that a typical element in $\mathfrak{n}_n(\mathbb{C})/R$ is of the form $\overline{A} = \{B \in \mathfrak{n}_n(\mathbb{C}) : B \text{ is similar to } A\}$ for some $A \in \mathfrak{n}_n(\mathbb{C})$. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we consider the nilpotent matrix

$$A_\lambda = \begin{bmatrix} A_{\lambda_1} & & \\ & \ddots & \\ & & A_{\lambda_k} \end{bmatrix} \in \mathfrak{n}_n(\mathbb{C}),$$

where for a positive integer m, A_m denotes the Jordan block with zero in the diagonal of size m, namely,

$$A_m = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix} \in \mathrm{Mat}_{m\times m}(\mathbb{C}).$$

For example, for $\lambda = (3,2) \vdash 5$, we have

$$A_\lambda = \begin{bmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & 0 & 0 \\ & & & 1 & 0 \end{bmatrix}$$

Problem 3. *Consider the map $\mathcal{P}(n) \to \mathfrak{n}_n(\mathbb{C})/R$ defined by $\lambda \mapsto A_\lambda$. Show that this map is a bijection.*

Given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$ (for simplicity, we assume that $F = \mathbb{C}$). Let c be an eigenvalue of A, we have defined algebraic multiplicity and geometric multiplicity of A at the eigenvalue c. We lack standard notations here. Recall that the geometric multiplicity of A at c is defined to be $\dim \mathrm{Ker}(A-cI)$. Recall that if A and B are similar, then they have the same eigenvalues. Moreover, at each eigenvalue, the algebraic multiplicities (and geometric multiplicities) of A and B are the same. Conversely, even if A and B have the same algebraic and geometric multiplicities at each eigenvalue, it does not mean that A and B are similar. A notation which generalize the geometric multiplicity is to consider $\dim(A-cI)^r$ for every positive integer r.

Problem 4.
(1) *Give two nilpotent matrices A, B such that A, B have the same geometric multiplicity, but A is not similar to B.*
(2) *Let $\lambda \vdash n$ be a partition of n. Determine $\dim \mathrm{Ker}(A_\lambda - 0I)^r = \dim \mathrm{Ker}(A_\lambda)^r$ in terms of λ (and r).*
(3) *Suppose that A, B are two nilpotent matrices (so the only eigenvalue of A, B is 0) such that $\dim \mathrm{Ker}(A)^r = \dim \mathrm{Ker}(B)^r$ for every r. Is it true that A is similar to B? In other words, if A is in Jordan canonical form, can the set $\{\dim \mathrm{Ker}(A)^r : r \geq 1\}$ uniquely determine the corresponding partition of A? If so, prove it. If not, give a counter example.*

For (2) and (3), if it is hard, at least try the case when $n = 7$. This is related to Exercise 11, page 250. Note that a positive answer of (3) implies the following: Let $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $\dim \mathrm{Ker}(A-cI)^r = \dim \mathrm{Ker}(B-cI)^r$ for any $c \in \mathbb{C}$ and for any positive integer r, then A and B are similar.

Problem 5. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that $(A^2+1)^2(A^2+2) = 0$. Find a relatively simple matrix in the conjugacy class of A.*

Hint: This is an exercise from class. You should know what I mean.

We consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where p is a prime integer. For simplicity, we assume that $p > 2$. It is known that there is an element $\kappa \in \mathbb{F}_p^\times = \mathbb{F}_p - \{0\}$ such that $x^2 - \kappa = 0$ has no solution in $\mathbb{F}_p$. For example, if $p = 3$, we can take $\kappa = 2$; if $p = 5$, we can take $\kappa = 2$ or 3. Such κ is not unique in general. But if κ_1, κ_2 are two such numbers, then $x^2 - \kappa_1\kappa_2^{-1} = 0$ has a solution in $\mathbb{F}_p$. For example, if $p = 5$, $\kappa_1 = 2, \kappa_2 = 3$, then $\kappa_1\kappa_2^{-1} = 4$ and $x^2 - 4 = 0$ has a solution in $\mathbb{F}_p$.

Problem 6. *Fix an element $\kappa \in \mathbb{F}_p^\times$ such that $x^2 - \kappa = 0$ has no solution in $\mathbb{F}_p$. Show that any element $g \in \mathrm{GL}_2(\mathbb{F}_p)$ is similar to one of the following type matrices*

(1) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \in \mathbb{F}_p^\times$;
(2) $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, a \in \mathbb{F}_p^\times$;
(3) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in \mathbb{F}_p^\times, a \neq b$;
(4) $\begin{bmatrix} a & b\kappa \\ b & a \end{bmatrix}, a \in \mathbb{F}_p, b \in \mathbb{F}_p^\times$.

Problem 7.
(1) *Try to classify conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_p)$.*
(2) *Try to classify conjugacy classes of $\mathrm{GL}_3(\mathbb{C})$ and $\mathrm{GL}_3(\mathbb{R})$.*