HOMEWORK 4

Due date: Next Monday

Exercises: 2, 3, 4, pages 261-262, Hoffman-Kunze,
Exercises: 1, 2, 3, page 269.
Exercises: 5, 7, 8, 9, 12, 13, 14, 16, 17. pages 276-277.

Problem 1. *Consider the matrix*

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q}).$$

Find the Smith normal form of $xI_3 - A$, the invariant factors of A and the rational canonical form of A. Determine if A has a Jordan canonical form. If so, find its Jordan canonical form.

Problem 2. *Consider the matrix*

$$A = \begin{bmatrix} x-1 & & \\ & (x-1)^2 & \\ & & x-2 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q}[x]).$$

Find the Smith normal form of A.

Problem 3. *Let F be a general field and $A \in \mathrm{Mat}_{n\times n}(F)$. Show that A is similar to A^t.*

Problem 4. *Let $\alpha = \sqrt[3]{2}$. Let $F = \left\{a + b\alpha + c\alpha^2 | a, b, c \in \mathbb{Q}\right\}$. We view F as a dimension 3 vector space over $\mathbb{Q}$. Let $\mathcal{B} = [1, \alpha, \alpha^2]$, which is an ordered basis of F over $\mathbb{Q}$. Given an element $x \in F$, we consider the linear operator $T_x : F \to F$ defined by $T_x(y) = xy$. Compute the matrix $A_x = [T_x]_{\mathcal{B}} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q})$ for $x = a + b\alpha + c\alpha^2$. Show that T_x is a semi-simple operator when F is viewed as vector space over $\mathbb{Q}$.*

Problem 5. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ be a matrix such that $A^3 - 2I_n = 0$, where $I_n \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ is the identity matrix. Show that $3|n$ (3 divides n). Write $n = 3k$ for a positive integer k. Show that A is similar the matrix*

$$\begin{bmatrix} & & 2I_k \\ I_k & & \\ & I_k & \end{bmatrix}.$$

Problem 6. *Consider the matrix*

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{Q}).$$

Find a semisimple matrix $S \in \mathrm{Mat}_{4\times 4}(\mathbb{Q})$ and a nilpotent matrix $N \in \mathrm{Mat}_{4\times 4}(\mathbb{Q})$ such that $A = S + N$ and $SN = NS$.

Hint: you can repeat the proof of Theorem 13, page 267, in this special case. It is useful to notice that the characteristic polynomial χ_A of A is $(x^2-2)^2$. Moreover, $x^2 - 2$ is irreducible over $\mathbb{Q}$. Here a matrix S is called semi-simple if the linear operator defined by S is semi-simple, or equivalently, if the minimal polynomial μ_S of S is a product of irreducible polynomials of multiplicity one.

Problem 7. *Let $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$. Define a map $(\ |\ ) : V \times V \to \mathbb{C}$ by*

$$(A|B) = \mathrm{tr}(AB^*).$$

Check that $(\ |\ )$ is an inner product on V.

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Let $T : V \to V$ be a linear operator. Assume that $\mu_T = q_1^{r_1} \dots q_k^{r_k}$ with q_i irreducible and pair wise distinct. To find a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is simpler, we usually do primary decomposition first and then apply cyclic decomposition to each component. More precisely, we have the primary decomposition

$$V = W_1 \oplus \cdots \oplus W_k,$$

with $W_i = \ker(q_i(T)^{r_i})$. Let $T_i : W_i \to W_i$ be the restriction of T to W_i. Then we have $\mu_{T_i} = q_i^{r_i}$. To find a simpler form of T_i, we need to find the invariant factors of T_i. To do so, we need to do the primary decomposition first, find the matrix $A_i = [T]_{\mathcal{B}_i}$ of T_i explicitly and then find the Smith norm form of $xI - A_i$. This seems very complicate. Now a question is: is it possible to read the invariant factors of T_i directly from the invariant factors of T? The answer is yes. Suppose that $p_1, \dots, p_m$ are invariant factors of T. Then we have $p_1 = \mu_T = q_1^{r_1} \dots q_k^{r_k}$. Since $p_i | p_1$ for $i \geq 2$, we can assume that

$$\begin{aligned} p_1 &= q_1^{s_{11}} q_2^{s_{12}} \dots q_k^{s_{1k}}, \\ p_2 &= q_1^{s_{21}} q_2^{s_{22}} \dots q_k^{s_{2k}}, \\ &\cdots \\ p_m &= q_1^{s_{m1}} q_2^{s_{m2}} \dots q_k^{s_{mk}}, \end{aligned}$$

where s_{ij} are non-negative integers with $s_{ij} \geq s_{i+1,j}$, and $s_{1j} = r_j$.

Problem 8. (1) *Given $f, g \in F[x]$ and $gcd(f, g) = 1$. Show that there is an isomorphism*

$$F[x]/(fg) \cong F[x]/(f) \times F[x]/(g),$$

which also preserves the product. (Recall that $F[x]/(f)$ is an F-algebra, which means that it is not only a vector space over F, there is also a product defined on $F[x]/(f)$.)

(2) *Using the above and the uniqueness of cyclic decompositions, show that the invariant factors of T_j are*

$$q_j^{s_{1j}}, q_j^{s_{2j}}, \dots, q_j^{s_{mj}}.$$

It is possible that many s_{ij} in the above sequence are zero and thus the corresponding term can be disregarded.

Hint for (2): Recall that the cyclic space $Z(\alpha; T)$ can be identified with $F[x]/(p_\alpha)$, where p_α is the annihilator of α. Use part (1) and the uniqueness of cyclic decompositions.

Problem 9. *Let $T : V \to V$ be a linear operator such that its invariant factors are given by*

$$(x - 2)^4(x + 1), (x - 2)^2(x + 1), (x - 2).$$

Find the corresponding invariant factors of T_1 and T_2, where $W_1 = \ker(T - 2I)^4, W_2 = \ker(T + I)$ and $T_i : W_i \to W_i$ is the corresponding linear operator defined by T_i. Moreover find the Jordan canonical form of T.