HOMEWORK 5

Due date: Monday of Week 6

Exercises: 1, 2, 3, 6, 11, 14, pages 288-290
Exercises: 1, 2, 4, 5, 6, 7, 10, 12, pages 298-299.

Problem 1. *Let m be any fixed positive integer. Show that*

$$\sum_{n=1}^{m} \frac{1}{n^2} \leq \frac{\pi^2}{6}, \text{ and } \sum_{n=1}^{m} \frac{1}{n^4} \leq \frac{\pi^4}{90}.$$

Try to generalize the above to $\sum_{n=1}^{m} \frac{1}{n^{2k}}$ for $k = 3, 4$.

Let $F = \mathbb{R}$ or $\mathbb{C}$. Let $(V, +, \cdot)$ be a vector space over F, where $+ : V \times V \to V$ is the addition and $\cdot : F \times V \to V$ is the scaler product. Define a new vector space $(\overline{V}, +, \odot)$ by

- $(\overline{V}, +) = (V, +)$ as an abelian group;
- the new scaler product $\odot : F \times V \to V$ is defined to be $c \odot \alpha := \overline{c} \cdot \alpha$, for all $c \in F, \alpha \in V$.

Note that, if $F = \mathbb{R}$, then $\overline{V}$ is the same as V.

Problem 2. (1) *Show that $(\overline{V}, +, \odot)$ is indeed a vector space.*
(2) *If $\dim V$ is finite, then $\dim \overline{V} = \dim V$.*
(3) *Let $T : V \to V$ be a linear map, show that $T : \overline{V} \to \overline{V}$ is also a linear map.*

Problem 3. *In Theorem 7, page 293 of the textbook, we only defined adjoint for linear operators $T \in \mathrm{End}(V)$. Try to generalize this concept to general linear maps, namely, try to define the adjoint T^* for $T \in \mathrm{Hom}(V, W)$, where V, W are two (possibly different) inner product spaces over F. Moreover, show that the adjoint you defined above indeed exists.*

Problem 4. *Let $A \in \mathrm{Mat}_{m\times n}(F)$. We consider the linear operator $T_A : F^n \to F^m$ by $T_A(\alpha) := A\alpha$. Here F^n and F^m are viewed as inner product space with respect to the standard inner product defined on them. Show that the adjoint of T_A is given by T_{A^*}, where $A^* = \overline{A^t}$ and the adjoint of T_A is defined in the last problem.*

We consider the column vector space $\mathbb{R}^n$ over $\mathbb{R}$. Let $(\ |\ )$ be the standard inner product on $\mathbb{R}^n$. Recall that $(x|y) = y^t x$, where y^t denotes the transpose of y. Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, we define the linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(x) = Ax$. Denote $\mathrm{Ker}(A) = \{x \in \mathbb{R}^n : Ax = 0\} = \mathrm{Ker}(T_A)$. Let $\mathrm{Row}(A)$ denote the space spanned by rows of A.

Problem 5. *Given $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Show that*
(1) $\mathrm{Ker}(A) = \mathrm{Ker}(A^t A)$;
(2) $\mathrm{rank}(A) = \mathrm{rank}(A^t A)$;
(3) $\mathrm{Row}(A) = \mathrm{Row}(A^t A)$.

Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional inner product space over F and let W be a subspace of V. We then have the orthogonal decomposition $V = W \oplus W^{\perp}$. Let $Proj_W : V \to V$ denotes the projection from V to W corresponding to this decomposition, namely, $Proj_W(\alpha, \beta) = \alpha$ for $\alpha \in W, \beta \in W^{\perp}$.

Problem 6. *Let $V = \mathbb{R}^n$ endowed with the standard inner product. Let $W \subset V$ be a subspace of V of dimension m. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ be a basis of W and consider the matrix*

$$M_{\mathcal{B}} = [\alpha_1, \ldots, \alpha_m] \in \mathrm{Mat}_{n\times m}(\mathbb{R}).$$

Here each α_i is a column vector.

(1) *Show that* $M_{\mathcal{B}}^t M_{\mathcal{B}} \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ *is invertible.*
(2) *Consider the matrix* $P_{\mathcal{B}} = M_{\mathcal{B}}(M_{\mathcal{B}}^t M_{\mathcal{B}})^{-1} M_{\mathcal{B}}^t \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. *Show that* $P_{\mathcal{B}}$ *is independent on the choice of* $\mathcal{B}$ *and thus it only depends on the space* W.
(3) *For any* $\alpha \in \mathbb{R}^n$, *show that* $P_{\mathcal{B}}\alpha \in W$. *(The notation* $P_{\mathcal{B}}\alpha$ *denotes the matrix product of* $P_{\mathcal{B}}$ *with* α*).*
(4) *Show that the map* $E : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $E(\alpha) = P_{\mathcal{B}}\alpha$ *is the same as the projection map* $Proj_W$. *(Hint: one way to do this is by choosing a good basis of* W *using the last part. Then compute the matrix* $P_{\mathcal{B}}$*.)*

Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $\beta \in \mathbb{R}^m$, we consider the linear system

$$Ax = \beta. \tag{0.1}$$

The above equation does not always have a solution. If the above equation has no solution, we can consider the following approximating solution, which is called *least square solution*. A vector $\hat{x} \in \mathbb{R}^n$ is called a least square solution of (0.1) if

$$||\beta - A\hat{x}|| \leq ||\beta - Ax||, \forall x \in \mathbb{R}^n.$$

It is clear that if $x \in \mathbb{R}^n$ is a solution of (0.1), then it is also a least square solution.

Problem 7. *Given* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *and* $\beta \in \mathbb{R}^m$. *Consider*

$$\mathrm{Im}(A) = \{y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n\} \subset \mathbb{R}^m,$$

and

$$\mathrm{Ker}(A^t) = \left\{y \in \mathbb{R}^m : A^t y = 0\right\} \subset \mathbb{R}^m.$$

(1) *Show that* $\mathrm{Im}(A)^\perp = \mathrm{Ker}(A^t)$. *Here* $\perp$ *is relative to the standard inner product on* $\mathbb{R}^m$.
(2) *Show that* $\hat{x} \in \mathbb{R}^n$ *is a least square solution of* (0.1) *if and only if* $\beta - A\hat{x} \in \mathrm{Im}(A)^\perp$ *if and only if* $A^t A\hat{x} = A^t\beta$.
(3) *Show that* (0.1) *always has a least square solution.*
(4) *Give a condition such that* (0.1) *has a unique least square solution.*

Problem 8. *Let* $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ *and* $\beta = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$. *Find a least square solution of the equation*

$$Ax = \beta.$$

Problems 5-8 were stated for the field $\mathbb{R}$. Try to consider the analogues for inner product spaces over $\mathbb{C}$. For example, in Problem 5, if we replace $\mathbb{R}$ by $\mathbb{C}$ and replace A^t by A^*, then show the same assertions hold. In Problem 6, what is the matrix $P_{\mathcal{B}}$ if the field is $\mathbb{C}$? Formulate and solve the least square solution problem over $\mathbb{C}$ as in Problem 7.

Problem 9. *Let* $F = \mathbb{R}$ *or* $\mathbb{C}$. *Let* $V = \mathrm{Mat}_{m\times n}(F)$ *and consider the map*

$$(\ |\ ) : V \times V \to F$$

defined by $(A \mid B) = \mathrm{tr}(AB^*)$, *where* $B^* = \overline{B}^t$. *Show that* $(\ |\ )$ *is an inner product on* V.

Given a matrix $A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, consider the Hilbert-Schimidt norm which is defined by

$$\|A\|_{HS} = \sqrt{\mathrm{tr}(AA^*)} = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Problem 10. *Show that*

$$\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}$$

for all $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.