HOMEWORK 6

Due date: Monday of Week 7

Exercises: 2, 6, 7, 10, 11, pages 308-311
Exercises: 2, 3, 8, 9, 11, 12, 13, 14, pages 317-318,
Exercises: 2, 3, 6, pages 347.

In this week, we focused on normal matrix (normal operators) on complex vector spaces. Similar results could be proved over real vector spaces once it is known that the corresponding eigenvalues are real numbers.

Problem 1. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be symmetric, namely, $A = A^t$.*

(1) *Let $a \in \mathbb{C}$ be a root of $\chi_A = \det(xI_n - A)$ (namely, a is a complex eigenvalue of A). Show that $a \in \mathbb{R}$.*
(2) *Show that there is an orthogonal matrix $P \in O_n(\mathbb{R})$ such that $P^T AP$ is diagonalizable.*

This was proved in class. Do it again.

Our treatment of the spectral decomposition theorem of normal operators over $\mathbb{C}$ was taken from Artin's book, and it is a little bit different from the treatment given in the textbook. In particular, Theorem 20, 21 were not covered in class. They are given in the following problem.

Problem 2.
(1) *Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T \in \mathrm{End}(V)$. Show that there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is upper triangular. (This is Theorem 21, page 316.)*
(2) *Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that there is a unitary matrix $P \in \mathrm{U}(n)$ and an upper triangular matrix U such that $A = PUP^{-1}$. This is called the **Schur decomposition** of A.*
(3) *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be normal. If $A = PUP^{-1}$ is the Schur decomposition of A, then U is diagonal.*

Remark 1. We have learned that any matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is triangulable; Schur decomposition says that A is triangulable by a unitary matrix.

Let $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$ and define $(A|B)_{\mathbb{C}} = \mathrm{tr}(B^*A)$. We have seen that $(\ |\ )_{\mathbb{C}}$ define an inner product on V when we view V as a vector space over $\mathbb{C}$.

Problem 3.
(1) *Given $A \in V = \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that any eigenvalue of A^*A is non-negative. (The eigenvalues of A^*A are real because A^*A is self-adjoint).*
(2) *Show that $Sp(A^*A) = Sp(AA^*)$. Here $Sp(A^*A)$ denotes the set of all eigenvalues of A^*A. Similarly, $Sp(AA^*)$ is the set of all eigenvalues of AA^*. An element in $Sp(A^*A)$ is called a singular value of A.*
(3) *If $Sp(A^*A) = \{0\}$, show that $A = 0$.*

(Hint: Since A^*A is self-adjoint (in particular, normal), there exists an orthonormal basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ of $\mathbb{C}^n$ such that each α_i is an eigenvector. Let λ_i be the corresponding eigenvalue, then $A^*A\alpha_i = \lambda_i\alpha_i$. Then consider $(A\alpha|A\alpha) = (\alpha|A^*A\alpha) = \ldots$. Here $(\ |\ )$ is the standard inner product on $\mathbb{C}^n$, not the one on V. We just lack enough notations.)

This problem gives another way to check $\mathrm{tr}(A^*A) = (A|A)_{\mathbb{C}} = 0$ implies $A = 0$. We now view V as a vector space over $\mathbb{R}$ (and write it as $V_{/\mathbb{R}}$ to emphasize that it is a vector space over $\mathbb{R}$) and define $(A|B)_{\mathbb{R}} = \mathrm{Re}((A|B)_{\mathbb{C}})$. Then $(\ |\ )_{\mathbb{R}}$ defines an inner product on $V_{/\mathbb{R}}$. This is easy to check. (Check it!)

Problem 4. *Let $V_{/\mathbb{R}} = \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the real vector space endowed with the inner product $(\ |\ )_{\mathbb{R}}$ defined above. Let*

$$W = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) | A = A^*\} \subset V.$$

(1) *Show that W is an $\mathbb{R}$ subspace of $V_{/\mathbb{R}}$ but not a $\mathbb{C}$-subspace of V. Thus we have the orthogonal decomposition $V_{/\mathbb{R}} = W \bigoplus W^{\perp}$. Here $W^{\perp}$ is of course defined w.r.t $(\ |\ )_{\mathbb{R}}$.*

(2) *Given $A \in V$, we write $A = A_1 + A_2$, with*

$$A_1 = \frac{A + A^*}{2}, A_2 = \frac{A - A^*}{2}.$$

Show that $A_1 \in W$ and $A_2 \in W^{\perp}$. Thus the orthogonal projection of $V_{/\mathbb{R}}$ to W associated to with the above orthogonal decomposition is $Proj_W(A) = \frac{1}{2}(A + A^*)$.

This problem might be helpful for solving Exercise 8, page 317.

Problem 5. *Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Suppose that $AA^* = A^2$. Show that A is self-adjoint.*

(Hint: Use the decomposition in Problem 4.)

1. Normal operators over $\mathbb{C}$

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Then the following are equivalent.

(1) A is normal, i.e., $AA^* = A^*A$;
(2) $(A\alpha|A\beta) = (A^*\alpha|A^*\beta)$, for any $\alpha, \beta \in \mathbb{C}^n$, where $(\ |\ )$ denotes the standard inner product on $\mathbb{C}^n$;
(3) $||A\alpha|| = ||A^*\alpha||$, for any $\alpha \in \mathbb{C}^n$;
(4) there exists a unitary matrix $P \in \mathrm{U}(n)$ and a diagonal matrix D such that $A = PDP^{-1}$;
(5) A_1 commutes with A_2, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2}$;
(6) there exists a polynomial $f \in \mathbb{C}[x]$ such that $A^* = f(A)$;
(7) let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, then $\mathrm{tr}(A^*A) = \sum_{j=1}^n |\lambda_j|^2$;
(8) $A^* = AP$ for some unitary matrix $P \in \mathrm{U}(n)$. (Actually, the condition $A^* = AP$ for $P \in \mathrm{U}(n)$ forces $AP = PA$.)

There are several other equivalent conditions, see this wiki-page if you can. See also Page 347, Exercise 4. Most of these were proved in classes and HW problems, see Ex 8, 13, page 317-318 for (5) (6). The above equivalences are for normal matrices. There is a similar version for normal operators. Try to translate the above equivalences to normal operators over $\mathbb{C}$. Normal operators over $\mathbb{R}$ is a little bit harder. We will learn them in next Chapter.

Problem 6. (1) *Prove the equivalence of (1) and (7).*
(2) *Prove the equivalence of (1) and (8).*

Hint: For the equivalence of (1) and (7) use Schur decomposition. This is easy. For the equivalence of (1) and (8), first show that the condition $A^* = AP$ for $P \in \mathrm{U}(n)$ implies $AP = PA$.

Let V be a finite dimensional vector space over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. Let $T \in \mathrm{End}(V)$ be a diagonalizable normal operator with spectral decomposition

$$T = c_1E_1 + \cdots + c_kE_k.$$

Recall that each E_i is a polynomial of T. In fact, we have

$$E_j = \prod_{i\neq j} \frac{T - c_iI}{c_j - c_i}$$

see Corollary, page 336. Let $S = \{c_1, \dots, c_k\}$ and $f : S \to F$ be a function. Then we define

$$f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k.$$

In general, the expression $f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k$ need not to be the spectral decomposition of $f(T)$, because it is possible that $f(c_i) = f(c_j)$ even $i \neq j$. Consider the simple example when $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $f(x) = x^2$ for $x \in \{-1, 1\}$. In this case $f(T) = E_1 + E_2$ is not the spectral decomposition of $f(T)$. If f is injective, then $f(T) = f(c_1)E_1 + \cdots + f(c_k)E_k$ is the spectral decomposition of $f(T)$. (Check this!) In particular, E_i is also a polynomial of $f(T)$. This can be checked directly as in the following problem.

Problem 7. *Let $f : S \to F$ be an injective map. Show that*

$$E_j = \prod_{i \neq j} \frac{f(T) - f(c_i)I}{f(c_j) - f(c_i)}$$

by a direct computation.

Problem 8. *Let $f : S \to F$ be an injective map. Let $U \in \mathrm{End}(V)$ be another linear operator. Show that U commutes with T if and only if U commutes with $f(T)$.*

Problem 9. *Let V be a finite dimensional inner product space over F with $F = \mathbb{R}$ or $\mathbb{C}$. Let T be a nonnegative operator on V and N be the unique non-negative operator on V such that $T = N^2$. Let $U \in \mathrm{End}(V)$. Show that U commutes with N if and only if U commutes with T.*

This is a special case of Problem 8.