# HOMEWORK 7

Due date: Monday of Week 8

You might assume the field is $\mathbb{C}$ in the exercises of the book if it helps.
Exercises: 6, 7, 8, 9, pages 324-325.
Exercises: 4, 6, 9, 10, 11, 12, 13, 14, 15, page 331-332

Exercise 4 of page 347 collects some equivalent definitions of normal operators. Many of them were proved in class and in last HW. In the next problem, show the equivalence of (a) and (i).

**Problem 1.** *Let $V$ be a finite dimensional inner product space over $F$ with $F = \mathbb{R}$ or $\mathbb{C}$. Let $T \in \mathrm{End}(V)$. Show that $T$ is normal if and only $UN = NU$, where $N$ is non-negative and $U$ is unitary such that $T = UN$, namely, $T = UN$ is the polar decomposition of $T$.*

This is a consequence of Problem **??**.

**Problem 2.** *For any $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, show that there exists $P_1, P_2 \in \mathrm{O}_n(\mathbb{R})$ and a diagonal matrix $D$ such that $A = P_1 D P_2$.*

This is called the singular decomposition of $A$. We talked this in class. Repeat it here.

**Problem 3.** *Consider the matrix*

$$A = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}).$$

*Find a singular value decomposition of $A$.*

**Problem 4.** *Denote $\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : AA^t = I_n\}$ and $\mathrm{SO}_n(\mathbb{R}) = \{g \in \mathrm{O}_n(\mathbb{R}) : \det(g) = 1\}$. We assume that $n$ is even, namely $n = 2m$ for a positive integer $m$. Given $A \in \mathrm{O}_n(\mathbb{R}) \setminus \mathrm{SO}_n(\mathbb{R})$ (this means $A \in \mathrm{O}_n(\mathbb{R})$ but $A \notin \mathrm{SO}_n(\mathbb{R})$).*

1. *If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, show that $\lambda\overline{\lambda} = 1$ and $\overline{\lambda}$ is also an eigenvalue of $A$ with the same algebraic multiplicity.*
2. *Show that $-1$ must be an eigenvalue of $A$ and $\dim_{\mathbb{R}} E_A(-1)$ is odd. Here $E_A(-1) = \{\alpha \in \mathbb{R}^n : A\alpha = -\alpha\}$ is the eigenspace corresponding to $-1$.*
3. *Let $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a matrix such that $AB = BA$. Show that $E_A(-1)$ is invariant under the left multiplication by $B$ and $B$ must have a real eigenvalue.*
4. *Consider the matrix $P = \begin{bmatrix} & -I_m \\ I_m & \end{bmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. If $Q \in \mathrm{O}_n(\mathbb{R})$ such that $QP = PQ$, show that $\det(Q) = 1$.*

Hint for (2): you can use the fact that $\dim_{\mathbb{C}} E_A(-1)_{\mathbb{C}} = \dim_{\mathbb{R}} E_A(-1)$, where $E_A(-1)_{\mathbb{C}} = \{\alpha \in \mathbb{C}^n : A\alpha = -\alpha\}$. This fact was proved in the solution of Ex section 7.2. You don't have to prove this fact again.

**Problem 5.** *Let $F = \mathbb{R}$ or $\mathbb{C}$ and $V$ be a finite dimensional inner product space over $F$. Given $T \in \mathrm{End}(V)$. Show that $T$ is normal if and only if there is a polynomial $f \in F[x]$ such that $T^* = f(T)$.*

## 1. Normal operators over $\mathbb{R}$

Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Note that the normal operator $T$ on $V$ is more complicate because $\chi_T$ is not necessary a product of linear factors, and thus $T$ is not (orthogonally) diagonalizable over $\mathbb{R}$ in general. Actually, we have seen that a normal linear operator $T$ is orthogonally diagonalizable if and only if it is self-adjoint. But many other properties of normal vectors still hold. Actually, the following are equivalent

(1) $T$ is normal, i.e., $TT^* = T^*T$;
(2) $(T\alpha|T\beta) = (T^*\alpha|T^*\beta)$, for all $\alpha, \beta \in V$;
(3) $||T\alpha|| = ||\alpha||$, for every $\alpha \in V$;
(4) $T_1$ commutes with $T_2$, where $T_1 = \frac{T+T^*}{2}, T_2 = \frac{T-T^*}{2}$;
(5) $T^* = TU$ for some orthogonal operator $U \in \mathrm{End}(V)$, where $U$ is said to be orthogonal if $UU^* = I$ and $U$ is a linear operator on real vector space;
(6) $U$ commutes with $N$, where $T = UN$ is the polar decomposition of $T$ with $N$ non-negative and $U$ orthogonal;
(7) there exists a polynomial $f \in \mathbb{R}[x]$ such that $T^* = f(T)$.

Many of the above equivalences were proved in class; the proofs of the rest are similar to the complex case. Notice the difference between real and complex case. Over the complex field, for a normal operator $T$, there exists a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal. Over the real field $\mathbb{R}$, this statement is no longer true. The simplest form of $[T]_{\mathcal{B}}$ is given in Theorem 17 and 18.

**Problem 6.** *Show that* (1) *is equivalent to* (5) *above.*

**Problem 7.** (1) *Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}).$$

*Chek that $A$ is normal. Moreover, find a polynomial $f \in \mathbb{R}[x]$ such that $A^t = f(A)$.*

(2) *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by the matrix $A$. Find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $[T]_{\mathcal{B}}$ is of the form*

$$\begin{bmatrix} c & & \\ & a & -b \\ & b & a \end{bmatrix},$$

*for $a, b, c \in \mathbb{R}$.*