HOMEWORK 8

Due date: Monday of Week 9

Exercises: 2, 4, 5, 6, 7, 9, 10, 11, pages 366-367
Exercises: 3, 6, 7, 8, 17, pages 373-375.

For problem 7 of page 373, go through the proof of Theorem 3, page 369. Of course you can orthogonal diagonolize the corresponding symmetric matrix. But here, try to practice the procedure given in the proof of Theorem 3.

Problem 1. *Let n be a fixed integer and let $V_n = \{f \in \mathbb{R}[x] \mid \deg(f) \le n\}$. Show that there exists a polynomial $p \in V_n$ such that for any $f \in V_n$, we have*

$$\int_0^1 f(x)p(x)dx = \int_0^1 f(x)\cos(x)dx.$$

If $n = 2$, find one such polynomial.

Problem 2. *Let p be an odd prime integer and let $\mathbb{F}_p$ be the finite field with p elements. Classify matrices in $\mathrm{GL}_n(\mathbb{F}_p)$ up to congruent. In other words, describe the equivalence classes of $\mathrm{GL}_n(\mathbb{F}_p)/\sim$, where the equivalence relation $\sim$ is congruent. How many equivalence classes are there?*

One can also ask: in each equivalence class, how many elements in that class? If you think this is hard, do it for $n = 2$.

Problem 3. *How many equivalence classes are there in $\mathrm{Mat}_{n\times n}(\mathbb{R})/\sim$? Here $\sim$ is still the congruent relation.*

Classification of quadratic forms over vector spaces over $\mathbb{Q}$ is very complicate. If you are interested in this, check Serre's book "a course in arithmetic". This is a subject in number theory.

Problem 4. *Let F be a field such that $char(F) \neq 2$. Let V be a finite dimensional vector space over F and let $f : V \times V$ be a symmetric bilinear form on V. Let W be a subspace of V and let $\{\alpha_1, \dots, \alpha_m\}$ be a basis of W. Consider the matrix $A = (A_{j,k})$ with*

$$A_{j,k} = f(\alpha_k, \alpha_j).$$

Show that A is invertible iff $W \cap W^\perp = 0$. If A is invertible, show that $V = W \oplus W^\perp$.

This is Theorem 7, page 332. Try to do it using what we learned in this week. Don't copy the proof given in page 332.

Let F be a field such that $char(F) \neq 2$. Let V be a finite dimensional vector space over F and let $f : V \times V$ be a symmetric or skew-symmetric bilinear form on V. A subspace W of V is called totally isotropic if $f(\alpha, \alpha) = 0$ for any $\alpha \in W$. Similarly, W is called anisotropic if there exists $\alpha \in W$ such that $f(\alpha, \alpha) = 0$.

Problem 5. *Let $f : V \times V \to F$ be a non-degenerate skew-symmetric bilinear form on V. Let W be a totally isotropic subspace of V. What is the maximal value of $\dim(W)$?*

1. Various decompositions

It is almost the end of linear algebra part of this course. It is a good time to review what we have learned about various decompositions of matrices, which are important parts of linear algebra. I will remind you those decompositions in the following, and it is helpful to keep a record of a proof for each decomposition. We learned all of these in class or from HW problems.

1.1. **Bruhat decomposition.** Let F be a general field. Denote by B_n the subset of $\mathrm{GL}_n(F)$ consisting of upper triangular invertible matrices with entries in F. Let $W \subset \mathrm{GL}_n(F)$ be the subset of permutation matrices. The set W was denoted by P in our previous HW. Recall that a matrix $g \in \mathrm{GL}_n(F)$ is called a permutation matrix if in each row and each column of g, there is only one nonzero term and that nonzero term is 1. Also recall that, we have a map

$$S_n \to W$$

$$\sigma \mapsto g_\sigma = \left[e_{\sigma(1)}, \dots, e_{\sigma(n)}\right],$$

where S_n is the symmetric group on n-elements which consists of bijections $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, e_i is the column vector whose only nonzero entry is 1 and it is at the i-th position. See HW 3 and HW 10 of last year. Recall that we have

$$g_{\sigma\tau} = g_\sigma g_\tau.$$

We also consider the special element $w_\ell \in W$ defined by

$$w_\ell = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

Problem 6. *Let A be a upper triangular matrix. Show that $w_\ell A w_\ell$ is lower triangular.*

Proposition 1 (Bruhat decomposition)**.** *For any element $g \in \mathrm{GL}_n(F)$, there exists $b_1, b_2 \in B$ and $w \in W$ such that $g = b_1 w b_2$. The elements b_1, b_2 are not unique in general, but $w \in W$ is uniquely determined by g. In other words, we have the decomposition*

$$\mathrm{GL}_n(F) = \coprod_{w \in W} BwB,$$

where $\coprod$ denotes disjoint union (which means $BwB \cap Bw'B = \emptyset$ if $w \neq w'$).

The decomposition $g = b_1 w b_2$ is equivalent to the LPU decomposition, which was in HW 3 of last year. We did not check the uniqueness in our HW.

Problem 7.
(1) *Show that the Bruhat decomposition in Proposition 1 is equivalent to the LPU decomposition in Problem 5, HW3 of last year.*
(2) *Prove the above Bruhat decomposition for $n = 2, 3, 4$ by proving the LPU decomposition first. Also check the uniqueness part for $w \in W$ in the decomposition for $n = 2, 3, 4$.*
(3) *Given $g \in \mathrm{GL}_n(F)$. Show that g has an LU decomposition (which means $g = g_1 g_2$ for g_1 lower triangular and g_2 upper triangular) if and only if all of its principle minors are all different from zero.*
(4) *Given $g \in \mathrm{GL}_n(F)$. Find a condition on g such that g has a decomposition $g = b_1 w_\ell b_2$ for $b_1, b_2 \in B$.*

Part (3) is a result from our textbook (Lemma, page 326). You don't need to submit a solution of this but you should know how to prove it. Part (3) is here because it gives you a hint for (4).

1.2. **C-R decomposition.**

Proposition 2 (C-R decomposition)**.** *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank r. Then there exists a matrix $C \in \mathrm{Mat}_{m\times r}(F)$ and a matrix $R \in \mathrm{Mat}_{r\times n}(F)$ such that $A = CR$.*

A special case of the above decomposition is when A has rank 1, then $A = uv$ for $u \in \mathrm{Mat}_{m\times 1}(F)$ and $v \in \mathrm{Mat}_{1\times n}(F)$. If $m = n$, then from $A = uv$, we can get that $A^2 = \mathrm{tr}(A)A$. The existence of the above C-R (which means column-row) decomposition was given in HW 5 of last year. Here is another related fact. Let k be a position integer with $k < r$, then there does not exist matrices $C \in \mathrm{Mat}_{m\times k}(F), R \in \mathrm{Mat}_{k\times n}(F)$ such that $A = CR$.

Problem 8. *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank r and let $A = CR$ be a C-R decomposition with $C \in \mathrm{Mat}_{m\times r}$ and $R = \mathrm{Mat}_{r\times n}$. For any $P \in \mathrm{GL}_r(F)$, if we denote $C' = CP \in \mathrm{Mat}_{m\times r}, R' = P^{-1}R \in \mathrm{Mat}_{r\times n}$, then $A = C'R'$ is another C-R decomposition. The question is: do we know all C-R decomposition has the above form? In other words, suppose that $A = CR = C'R'$ with $C, C' \in \mathrm{Mat}_{m\times r}, R, R' \in \mathrm{Mat}_{r\times n}$ such that*

$$A = CR = C'R'.$$

Is there a matrix $P \in \mathrm{GL}_r(F)$ such that $C' = CP$ and $R' = P^{-1}R$? If so, prove it. If not, find a counter-example.

This is certain uniqueness of C-R decomposition. If you think this hard, try to consider some examples with small m, n, r, for example, when $m = n = 3$ and $r = 2$.

1.3. Jordan decomposition.

Proposition 3 (Jordan decomposition)**.** *Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix such that μ_A is a product of linear factors. There exists a unique diagonalizable matrix $D \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $DN = ND$ and $A = D + N$.*

Moreover, we know that such D, N are polynomials of A. This is Theorem 13, page 222.

Proposition 4 (Jordan decomposition, semisimple version)**.** *Let F be a field of characteristic zero. Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix. Then there exists a unique semi-simple matrix $S \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $SN = NS$ and $A = S + N$.*

This is Theorem 13, page 267.

1.4. **Iwasawa decomposition.** Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let B_n be the upper triangular matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

Proposition 5 (Iwasawa decomposition)**.** *We have $\mathrm{GL}_n(F) = B_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists an element $b \in B_n$ and an element $k \in K_n$ such that $g = bk$.*

This is equivalent to Theorem 14 of page 305. Explain the equivalence between the above Proposition and Theorem 14 of page 305.

Problem 9. *Consider the matrix*

$$g = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{R}).$$

Find a matrix $b \in B_3$ and $k \in \mathrm{O}_3(\mathbb{R})$ such that $g = bk$.

1.5. **Singular value decomposition, polar decomposition and Cartan decomposition.** Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let A_n be the set of all diagonal matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

Proposition 6 (Cartan decomposition)**.** *We have $\mathrm{GL}_n(F) = K_n \cdot A_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists $k_1, k_2 \in K_n$ and $a \in A_n$ such that $g = k_1 a k_2$.*

This is just a slightly different way to say the singular value decomposition.

Proposition 7 (Polar decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists a matrix $k \in K_n$ and a positive matrix p such that $g = kp$.*

This is Theorem 14, page 342. Singular value decomposition and polar decomposition are closely related.

1.6. **Schur decomposition.** Let $F = \mathbb{C}$ and let $K_n = \mathrm{U}(n)$. Let $B_n \subset \mathrm{GL}_n(F)$ be the subset consisting of upper triangular matrices.

Proposition 8 (Schur decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists an element $k \in K_n$ and an element $b \in B_n$ such that $g = kbk^{-1}$.*

This is Theorem 21, page 316. See also HW 6.