HOMEWORK 1

Due date: Tuesday, Week 2,

Exercises: 1.6, 1.7, 1.8, 2.1, 2.2, 3.1, 3.3, 3.4, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 4.2 pages 354-355, Artin's book.

Any ring in our course is a commutative ring with 1.

Let R be a ring. An ideal $\mathfrak{p} \subset R$ with $\mathfrak{p} \neq R$ is called prime if for any $x, y \in R, xy \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Let $\operatorname{Spec}(R)$ be the set of all prime ideals of R, which is called the spectrum of R.

Problem 1. *Let R be a ring and $\mathfrak{p}$ be a prime ideal of R. Let I, J be any two ideals of R such that $IJ \subset \mathfrak{p}$, show that either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.*

Here the product ideal IJ is defined in Ex.3.13, page 355.

Problem 2. *Determine $\operatorname{Spec}(R)$, when R is (a) $\mathbb{Z}$, (b) $F[x]$ for a field F, (c) $\mathbb{Z}/6\mathbb{Z}$, (d) $F[x]/(x^2)$.*

Problem 3. *Let $\phi : A \to B$ be a ring homomorphism. Let $\mathfrak{p}$ be a prime ideal of B. Show that $\phi^{-1}(\mathfrak{p}) := \{x \in A : \phi(x) \in \mathfrak{p}\}$ is a prime ideal of A. Thus there is a map $\phi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ defined by $\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$.*

Problem 4. *Let R be a ring. An element $a \in R$ is called nilpotent if $a^k = 0$ for some integer $k \geq 1$. Let $\mathfrak{N}$ be the set of all nilpotent elements of R. Show that $\mathfrak{N}$ is an ideal. This ideal $\mathfrak{N}$ is called the nilradical of R. Moreover, show that $\mathfrak{N}$ is contained in every prime ideal of R.*

A ring is called reduced if its nilradical is zero.

Problem 5. *Determine the nilradical of the following rings:*

$$(1),\ \mathbb{Z}/4\mathbb{Z}; \quad (2),\ \mathbb{Z}/12\mathbb{Z}; \quad (3),\ \mathbb{Z}/n\mathbb{Z}.$$

Problem 6. *Let R be a ring and let I be an ideal. Define*

$$\sqrt{I} = \{x \in R : \exists n > 0, n \in \mathbb{Z}, \text{ s.t. } x^n \in I\}.$$

Show that $\sqrt{I}$ is an ideal of R. Let $\pi : R \to R/I$ be the projection map. Find an ideal $\mathfrak{b} \subset R/I$ such that $\sqrt{I} = \pi^{-1}(\mathfrak{b})$.

Problem 7. *Let R be a ring and $x \in I$. Denote*

$$\operatorname{Ann}(x) = \{r \in R : rx = 0\}.$$

Show that $\operatorname{Ann}(x)$ is an ideal of R. It is called the annihilator of x.

1. Zariski topology

There is no need to submit the rest part of the HW.

Reference for this section: Atiyah-MacDonald, commutative algebra, section 1. Actually, many exercises were taken from this book. You are encouraged to read this book and finish all the exercises from that book if you want to learn pure math in the future.

Let X be a set. Denote by $\mathcal{P}(X)$ the power set of X. By definition, $\mathcal{P}(X)$ is the set of all subsets of X. In other words, $\mathcal{P}(X) = \{A \mid A \subset X\}$. A different notation for $\mathcal{P}(X)$ is 2^X.

Definition 1. A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subset of $\mathcal{P}(X)$ satisfying the following conditions.

(1) $X \in \mathcal{T}, \emptyset \in \mathcal{T}$. Here $\emptyset$ denotes the empty set.
(2) Any finite intersection of sets in $\mathcal{T}$ is still in $\mathcal{T}$. More precisely, let I be a finite set and suppose that we are given an element $U_i \in \mathcal{T}$ for each $i \in I$. Then $\cap_{i \in I} U_i \in \mathcal{T}$.

(3) Any union of sets in $\mathcal{T}$ is still in $\mathcal{T}$. In other words, let J be any set and suppose that we are given a set $U_j \in \mathcal{T}$ for any $j \in J$. Then $\cup_{j\in J} U_j \in \mathcal{T}$.

Given a topological space $(X, \mathcal{T})$, $\mathcal{T}$ is called a topology on X. An element $U \in \mathcal{T}$ is called an open subset of X (with respect to the given topology $\mathcal{T}$). The above 3 conditions can be restated as: (1) X and the empty set are open sets; (2) Any finite number of intersections of open sets is still open; (3) Any union of open sets is still open.

Given a topology $\mathcal{T}$ on X. A subset $V \subset X$ is called closed if it is the complement of an open subset. One can also define topology using closed subsets.

You should learn from Calculus class that $\mathbb{R}^n$ has a natural topology, where a subset $U \subset \mathbb{R}^n$ is called open if it is union of open balls. Here an open ball is a set of the form $B_r(a) = \{x \in \mathbb{R}^n \mid |x - a| < r\}$ for a fixed $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$.

Let R be a ring and let $S \subset R$ be any subset. Define $V(S) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid S \subset \mathfrak{p}\}$.

Problem 8. *Let R be a ring. Show that*

(1) *If $\mathfrak{a}$ is the ideal generated by S, then $V(S) = V(\mathfrak{a})$. Here $\mathfrak{a}$ is called the ideal generated by S if it is smallest ideal that contains S. In other words, any element in $\mathfrak{a}$ is of the form $\sum r_i x_i$ with $r_i \in R, x_i \in S$. Here the sum is a finite sum.*
(2) $V(0) = \operatorname{Spec}(R), V(1) = \emptyset$.
(3) *Let I be a set and suppose that we are given a subset $S_i \subset R$ for each $i \in I$. Then $V(\cup_{i\in I} S_i) = \cap_{i\in I} V(S_i)$.*
(4) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$ *for any ideals $I, J \subset A$.*
(5) *Let $\mathcal{T} = \{\operatorname{Spec}(R) - V(S) \mid S \subset R\}$. Show that $\mathcal{T}$ is a topology on $\operatorname{Spec}(R)$. It is called the Zariski topology on $\operatorname{Spec}(R)$.*

Problem 9. *Let R be a ring. Describe all points $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\{\mathfrak{p}\} \subset \operatorname{Spec}(R)$ is closed in the Zariski topology.*

Problem 10. *Let R be a ring. Show that $\operatorname{Spec}(R)$ is compact.*

Here a topological space X is called compact if every open covering of X has a finite subcovering. More precisely, if $\{U_i\}_{i\in I}$ is a collection of open subsets of X such that $X \subset \cup_{i\in I} U_i$, then there exists a finite subset $J \subset I$ such that $X \subset \cup_{j\in J} U_j$.